

# *The Equation $Tx = y$ in a Reflexive Operator Algebra*

ALAN HOPENWASSER

In the study of reflexive operator algebras with commutative subspace lattices it is frequently of technical use to know of the existence of an operator in the algebra which maps a given vector to a second given vector. For example, Ringrose, in his initial study of nest algebras [2], makes effective use of the existence of rank one operators in the algebra. In a further study of nest algebras [1], Lance gives necessary and sufficient conditions on vectors  $x$  and  $y$  to guarantee the existence of an operator  $T$  in a given nest algebra such that  $Tx = y$ . In addition, Lance specifies the minimum norm such an operator can have. It turns out that Lance's Theorem is true in the wider context of reflexive algebras with commutative subspace lattices; it is the purpose of this note to prove Lance's Theorem in this generality.

To begin, suppose that  $\mathcal{L}$  is a commutative subspace lattice (*i.e.* a complete lattice of mutually commuting orthogonal projections which contains 0 and 1) acting on a Hilbert space  $\mathcal{H}$ . Let  $\mathcal{A} = \text{Alg } \mathcal{L}$  be the algebra of all bounded linear operators on  $\mathcal{H}$  which leave invariant each projection  $E$  in  $\mathcal{L}$ . Assume that  $x$  and  $y$  are vectors in  $\mathcal{H}$  and that  $S$  is an operator in  $\mathcal{A}$  such that  $Sx = y$ . Then

$$\begin{aligned}\|E^\perp y\| &= \|E^\perp Sx\| = \|E^\perp S E^\perp x\| \\ &\leq \|S\| \|E^\perp x\|, \text{ for all } E \in \mathcal{L}.\end{aligned}$$

(Here,  $E^\perp$  denotes  $1-E$ , the complement of  $E$ , and we use the invariance of  $E$  under  $S$  in the form  $E^\perp S = E^\perp S E^\perp$ .) If, for convenience, we adopt the convention that a fraction whose numerator and denominator are both zero is equal to zero, then the inequalities above may be stated in the form

$$\sup_{E \in \mathcal{L}} \frac{\|E^\perp y\|}{\|E^\perp x\|} \leq \|S\|.$$

The theorem which we shall prove is the following:

**Theorem.** *Let  $\mathcal{L}$  be a commutative subspace lattice acting on a Hilbert space  $\mathcal{H}$  and let  $\mathcal{A} = \text{Alg } \mathcal{L}$ . Let  $x$  and  $y$  be two vectors in  $\mathcal{H}$  and assume that*