

Approximately Finite C^ -Algebras and Continued Fractions*

EDWARD G. EFFROS & CHAO-LIANG SHEN

Introduction. A C^* -algebra A is said to be *approximately finite* (“AF”) if A is the norm-closure of an ascending union of finite-dimensional C^* -algebras A_m (see [1]). Elliott has reduced the (stable) classification of such algebras A to the classification of certain ordered abelian groups, denoted by $G(A)$ or $\tilde{K}_0(A)$, which he called dimension groups (see [4], [6] and below). Elliott showed that rank one dimension groups are completely determined by their group structure, and thus classifiable. Roughly speaking, this explains why Glimm and Dixmier were able to classify the UHF and matroid algebras respectively [9], [2].

In this paper we shall completely classify the *dicyclic* dimension groups, *i.e.*, those for which the underlying group is isomorphic to \mathbb{Z}^2 . We find that there are two isolated ordered groups, denoted by $\mathbb{Z} \oplus_{\text{ord}} \mathbb{Z}$ and $\mathbb{Z} \oplus_{\text{lex}} \mathbb{Z}$, and a family (\mathbb{Z}^2, P_α) indexed by positive irrationals modulo the action of $GL(2, \mathbb{Z})$. We then show that the continued fraction expansion of α explicitly determines AF algebras A with $G(A) \cong_{\text{ord}} (\mathbb{Z}^2, P_\alpha)$. Finally we show how one can use Bratteli diagrams to determine dicyclic dimension groups. We shall examine *polycyclic* dimension groups in a subsequent paper.

1. Some preliminaries. We use the notations \mathbb{N} , \mathbb{N}_0 , \mathbb{Z} for the positive, non-negative, and all the integers, respectively, \mathbb{Q} for the rationals, and \mathbb{R} , \mathbb{R}^+ for the reals and non-negative reals, respectively.

We let e_i ($1 \leq i \leq r$) be the canonical generators for \mathbb{Z}^r , and we define the *simplicial* ordering on \mathbb{Z}^r by the subset

$$(\mathbb{Z}^r)^+ = \left\{ \sum a_i e_i : a_i \in \mathbb{N}_0 \right\}.$$

We identify the homomorphisms $\varphi : \mathbb{Z}^s \rightarrow \mathbb{Z}^r$ with the matrices $[a_{ij}]$, $a_{ij} \in \mathbb{Z}$, where $\varphi(e_j) = \sum a_{ij} e_i$. Then φ is positive (*i.e.*, an “order homomorphism” in the terminology of [11]) if and only if $a_{ij} \in \mathbb{N}_0$ for all i, j . We let $GL(r, \mathbb{Z})$ be the algebraic isomorphisms $\varphi : \mathbb{Z}^r \rightarrow \mathbb{Z}^r$, *i.e.*, the group of unimodular matrices with determinant ± 1 , and $GL(r, \mathbb{Z})^+$ be the corresponding positive matrices.