

# *Reflection Methods in the Elastic-Plastic Torsion Problem*

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**1. Introduction.** Let  $Q$  be a simply connected domain in  $R^2$ , and let  $\mu$  be a positive number. Consider the variational inequality for a function  $u$ :

$$(1.1) \quad \int_Q \nabla u \cdot \nabla (v - u) dx \geq \mu \int_Q (v - u) dx \quad \forall v \in K, u \in K$$

where

$$K = \{w \in H_0^1(Q), |\nabla w(x)| \leq 1 \text{ a.e.}\}.$$

We assume that the boundary  $\partial Q$  of  $Q$  is piecewise  $C^3$ , and denote the vertices of  $\partial Q$  by  $V_i$  ( $1 \leq i \leq \tau$ ).

We review some well-known facts (for more details see [1]–[3], [5]–[7], [9], [10]).

The problem (1.1) has a unique solution  $u$ . Further,  $u \in W_{\text{loc}}^{2,\infty}(Q)$  and, for any subdomain  $Q_0$  of  $Q$  such that  $\partial Q_0 \cap \partial Q$  contains no vertices  $V_i$ ,  $u$  is in  $W^{2,p}(Q_0)$  for any  $p < \infty$ . The solution  $u$  coincides with the solution of the obstacle problem with  $d(x)$  as the obstacle from above;  $d(x)$  being the distance from  $x$  to  $\partial Q$ . That means that  $u$  solves (1.1) with  $K$  replaced by

$$K_0 = \{w \in H_0^1(Q), w(x) \leq d(x)\}.$$

The solution  $u$  is greater than or equal to zero. The sets

$$E = \{x \in \bar{Q}; |\nabla u(x)| < 1\}$$

$$P = \{x \in \bar{Q}; |\nabla u(x)| = 1\}$$

are called the *elastic* and *plastic* sets, respectively, and

$$E \cap Q = \{x \in Q; u(x) < d(x)\},$$

$$P \cap Q = \{x \in Q; u(x) = d(x)\}.$$

The set  $\partial E \cap Q$  is called the *free boundary*.

If a point  $x^0$  in  $Q$  is plastic, and if  $y^0$  is the nearest point to  $x^0$  on  $\partial Q$ ,