

On the Borel Subspaces of Algebraic Structures

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Introduction. Let G be a connected abelian Polish group which admits an element of infinite order. Theorem 1; for each ordinal α , $1 \leq \alpha < \omega_1$, there is a subgroup of G which is of exactly additive class α in G . For each ordinal, α , $2 \leq \alpha < \omega_1$, there is a subgroup of G which is of exactly multiplicative (ambiguous) class α in G . Theorem 2; if X is an infinite dimensional separable Banach space, then there are subspaces of X of exactly the same Borel classes as in the previous theorem.

Let us recall that the sets of multiplicative (additive) class 0 in a metric space are the closed (open) sets; the sets of multiplicative (additive) class α , $\alpha > 0$ are those sets which can be expressed as the intersection (union) of countably many sets of additive (multiplicative) class less than α ; the sets of ambiguous class α are the sets which are of both multiplicative and additive class α ; a set is of exactly multiplicative (additive) class α provided it is of that class but is not of additive (multiplicative) class α ; a set is of exactly ambiguous class α provided it is of ambiguous class α , but is not of any lower class [3].

Let us recall that a subset H of an abelian group G is said to be independent provided that if h_1, \dots, h_n are elements of H , b_1, \dots, b_n are integers and

$$\sum_{i=1}^n b_i h_i = e,$$

then $b_1 = b_2 = \dots = b_n = 0$.

The idea of the proof of Theorem 1 is first to obtain a compact, perfect, totally disconnected subset M of G which is independent (Theorem 0) and then to simply take a subset H of M of a given Borel class and show that $\langle H \rangle$, the subgroup of G generated by H , is of the same class.

Theorem 0. *Let G be a connected, abelian Polish group which has an element of infinite order. Then there is an independent compact, perfect, totally disconnected subset M of G .*

Proof. For each integer a , set