

A Note on Restriction

PETER A. TOMAS

Let φ be a positive, smooth function with support in $[-1, 1]$ and let $\varphi_\delta(\bar{x}) = \varphi(\delta^{-1}(|\bar{x}| - 1))$. Define multipliers on $L^p(\mathbf{R}^2)$ by $\widehat{T_\delta f}(\bar{\xi}) = \varphi_\delta(\bar{\xi})\hat{f}(\bar{\xi})$.

A precise bound on the operators T_δ was used by Cordoba [2] to analyze Bochner-Riesz means. Recently, Cordoba [3] has shown

$$(1) \quad \|T_\delta f\|_4 \leq c |\log \delta|^{1/4} \|f\|_4$$

where the norm estimate is optimal. From this estimate, Cordoba immediately derives the $L^4(\mathbf{R}^2)$ boundedness of Bochner-Riesz means, $(1 - |\bar{\xi}|^2)_+^\lambda$.

The purpose of this note is to establish a result which is in some sense dual, the estimate

$$(2) \quad \|T_\delta \hat{f}\|_4 \leq c \delta^{3/4} |\log \delta|^{1/4} \|f\|_4.$$

This estimate is optimal [4]. This is a “restriction” theorem, in the sense of Fefferman-Stein [5], as follows: $T_\delta \hat{f} = \hat{\varphi}_\delta * \hat{f}$. Dualizing, $\|\varphi_\delta \hat{f}\|_{4/3} \leq c \delta^{3/4} |\log \delta|^{1/4} \|f\|_{4/3}$, and $(1/\delta \int_{1-\delta}^{1+\delta} \int_{S'} |\hat{f}(re^{i\theta})|^{4/3} r \, dr \, d\theta)^{3/4} \leq c |\log \delta|^{1/4} \|f\|_{4/3}$. If the $\log \delta$ factor were not present, one could take the limit as $\delta \rightarrow 0$, and obtain $\|\hat{f}|_{S'}\|_{4/3} \leq c \|f\|_{4/3}$. This is known to be false, although the estimate

$$(3) \quad \|\hat{f}|_{S'}\|_{4/3} \leq c_p \|f\|_p \quad 1 \leq p < 4/3$$

was established by Fefferman and Stein [5]. Estimate (2) therefore measures the extent to which (3) fails at $p = 4/3$.

The statements of (1) and (2) hint at subtle interplay between Bochner-Riesz and restriction. Such a connection has been made precise by Fefferman [5] (see also the work of Stein in [6]) and by Carleson and Sjolin [1]. The actual proof of estimate (2) follows a new proof by Cordoba [4] of (3), which in turn has many elements in common with the proof in [3] of (1). The connection does not seem to go deeper; in particular it will become clear that in spite of formal similarities, (2) is substantially easier than (1).

I am grateful to Professor A. Cordoba for showing me his unpublished work, which was central to the results in this paper.

To prove (2) we decompose φ_δ into 4 pieces, each supported in one quadrant. Rotational symmetry shows that it suffices to prove (2) for the piece with