

An Addendum to Limsups of Lats

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As the title suggests, this note is a postscript to the preceding paper [PRH 3] of P. R. Halmos. This note should not (and probably cannot) be read before [PRH 3]. All of our notation and terminology is taken from [PRH 3]. Throughout, H denotes a separable infinite-dimensional Hilbert space.

The main result (Theorem 4) of this note concerns the points of continuity of Lat. The following theorem (see the corollary) shows that, in a sense, Halmos' "possible application" of the upper semicontinuity of Lat to subnormal operators really works. This result is not very satisfying, however, since its proof relies on Scott Brown's invariant subspace theorem for subnormal operators [SB]. Part (1) of this theorem appears in greater generality in [H, Lemma 7.10].

Theorem 1. *Suppose M is an infinite-dimensional subspace of H , let e_1, e_2, \dots be an orthonormal basis for M , and suppose $\{U_n\}$ is a sequence of unitary operators from M onto H such that $U_n e_k = e_k$ for $1 \leq k \leq n < \infty$. If $T(M) \subset M$, then*

(1) $U_n^* T U_n \rightarrow T|_M$ strongly,

(2) $\liminf_n \text{Lat } U_n^* T U_n = \text{Lat}(T|_M)$; here the limit is defined in terms of the convergence on $2^{\mathcal{S}}$ induced by the strong topology on \mathcal{S} .

Note that statement (2) above is equivalent to the assertion that $\liminf_n \text{Lat } U_n^* T U_n \supset \text{Lat}(T|_M)$ since the upper semicontinuity of Lat and (1) imply that $\limsup_n \text{Lat } U_n^* T U_n \subset \text{Lat}(T|_M)$.

Proof. To prove (1), first note that since $\{U_n^* T U_n\}$ is norm bounded, the strong convergence need be checked only on the basis vectors e_1, e_2, \dots . This reduces the proof of (1) to a computation that is sufficiently elementary to be left to the reader.

To prove (2) suppose that $T(M) \subset M$ and N is a subspace of M that is invariant for $T|_M$. If P is the projection of H onto N , then $P(M) \subset M$, and, by (1), $U_n^* P U_n \rightarrow P|_M$ strongly. Since $U_n^* P U_n \in \text{Lat } U_n^* T U_n$ for $n = 1, 2, \dots$, we conclude that $N \in \liminf_n \text{Lat } U_n^* T U_n$. Thus $\text{Lat}(T|_M) \subset \liminf_n \text{Lat } U_n^* T U_n$. By the remark preceding the proof, the proof of (2) is complete.