

Weighted Wirtinger and Poincaré Inequalities on Unbounded Domains

ALAN R. ELCRAT & H. ALAN MacLEAN

Introduction. This article is concerned with establishing Poincaré and Wirtinger type inequalities for integrals of functions and their derivatives over unbounded domains. It is well known that the inequality

$$\int_{\Omega} |u|^p dx \leq c \int_{\Omega} |\text{grad } u|^p dx$$

holds for a bounded domain if $u = 0$ on $\partial\Omega$ (Poincaré), or if $\int_{\Omega} u dx = 0$ (Wirtinger), but does not hold, for example, on an unbounded domain Ω with the property that the intersection of Ω with every ray is either empty or contains an infinite interval. We shall establish here inequalities of this type for integrals with respect to weight functions which satisfy certain conditions. For instance, we show that if Ω is the positive real axis, and ϕ is positive and decays exponentially as x tends to infinity, then there is a constant c such that

$$(1) \quad \int_0^{\infty} |u|^p \phi dx \leq c \int_0^{\infty} |u'|^p \phi dx$$

for any locally absolutely continuous function u which vanishes at the origin.

The outline of the paper is as follows. In the first section we give elementary proofs of inequalities of the type indicated in (1). The results that we present may be thought of as related to earlier work of Redheffer [2] and others (see the bibliography of [2]). However, the particular form that we give to these results is important in what follows. In the second section we establish inequalities for functions in weighted Sobolev spaces over unbounded domains in R^n . The inequalities obtained here are derived from the one-dimensional results by employing “spherical” coordinates in R^n . In particular, the n -dimensional analogue of (1) is established under reasonable hypotheses on ϕ . Finally, in the third section we prove a general theorem which requires minimal smoothness of $\partial\Omega$, and which has as special cases both Poincaré and Wirtinger type inequalities. The proof utilizes the results of Section Two and certain methods introduced recently by Meyers [3] and Meyers and Ziemer [4]