

$(\partial\bar{\partial})_b$ and the Real Parts of CR Functions

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1. Preliminaries. Let $\Gamma \subset \mathbb{C}^n$ be a smooth submanifold and let $\Lambda^{p,q}$ denote the smooth (p,q) -forms on \mathbb{C}^n . To study the action of $\partial\bar{\partial}$ tangential to Γ , we will take the induced boundary complex (cf. [1]). Locally we may find smooth functions r_1, \dots, r_d such that $\Gamma = \{r_1 = \dots = r_d = 0\}$ and $dr_1 \wedge \dots \wedge dr_d \neq 0$ on Γ . We let $\mathcal{I}^{p,q}$ be the ideal in $\Lambda^{p,q}$ generated by $\{r_j, \partial r_j, \bar{\partial} r_j, \partial\bar{\partial} r_j: 1 \leq j \leq d\}$. It follows that $\partial\bar{\partial}(\mathcal{I}^{p,q}) \subset \mathcal{I}^{p+1,q+1}$. If we set $\mathcal{E}^{p,q} = \Lambda^{p,q} |_{\Gamma} \pmod{\mathcal{I}^{p,q}}$, then the *tangential complex* $(\partial\bar{\partial})_{\Gamma}$ is defined as the map on the quotient

$$(\partial\bar{\partial})_{\Gamma}: \mathcal{E}^{p,q} \rightarrow \mathcal{E}^{p+1,q+1}$$

induced by $\partial\bar{\partial}$. Clearly, $\mathcal{E}^{0,0} = C^{\infty}(\Gamma)$. If $\mathcal{E}^{p,q}$ and $\mathcal{E}^{p+1,q+1}$ have constant dimension on Γ , they are vector bundles, and $(\partial\bar{\partial})_{\Gamma}$ is a system of second order differential operators.

If Γ is a hypersurface, then locally it is the boundary of an open set, and $(\partial\bar{\partial})_{\Gamma}$ in this case will also be called the *boundary complex* $(\partial\bar{\partial})_b$. When Γ is a hypersurface, $\mathcal{E}^{1,1}$ has dimension $(n-1)^2$ if $\partial r \wedge \bar{\partial} r \wedge \partial\bar{\partial} r = 0$ on Γ , and $(n-1)^2 - 1$ otherwise. If $n = 2$, we have $(\partial\bar{\partial})_b = 0$ unless $\partial r \wedge \bar{\partial} r \wedge \partial\bar{\partial} r = 0$, i.e. Γ is Levi flat. If Γ is Levi flat, then $\partial_b \bar{\partial}_b = (\partial\bar{\partial})_b$; otherwise, these two operators are different.

Γ is said to be a *CR manifold* if the dimension of $\mathcal{E}^{1,0}$ is constant on Γ . In this case, we may choose a subset of $\{r_1, \dots, r_d\}$, which we write as $\{r_1, \dots, r_e\}$, such that $\{\partial r_1, \dots, \partial r_e\}$ generates $\mathcal{I}^{1,0}$ locally and $\partial r_1 \wedge \dots \wedge \partial r_e \neq 0$. A simple computation shows that $u \in \Lambda^{p,q}$ satisfies $(\partial\bar{\partial})_{\Gamma} u = 0$ if and only if there are (p,q) -forms $\alpha_1, \dots, \alpha_d$ such that

$$\begin{aligned} (1) \quad & \partial r_1 \wedge \dots \wedge \partial r_e \wedge \bar{\partial} r_1 \wedge \dots \wedge \bar{\partial} r_e \wedge \partial\bar{\partial} u \\ & = \partial r_1 \wedge \dots \wedge \partial r_e \wedge \bar{\partial} r_1 \wedge \dots \wedge \bar{\partial} r_e \wedge \sum_{j=1}^d \alpha_j \wedge \partial\bar{\partial} r_j \end{aligned}$$

holds. It is immediate that if f is a CR function on Γ , then $(\partial\bar{\partial})_{\Gamma} \operatorname{Re} f = 0$; the point of this paper is to establish the converse of this fact.

Since we are only interested in $(\partial\bar{\partial})_{\Gamma}$ acting on functions (0-forms), we give another description of the condition $(\partial\bar{\partial})_{\Gamma} u = 0$. At a point $z \in \Gamma$ we may define a subspace of second order operators on Γ by