

Higher Integrability from Reverse Hölder Inequalities

E. W. STREDULINSKY

The following was motivated by the need for a generalization of a lemma of F. W. Gehring [1]. The accidental deletion of the hypothesis “ $g = 0$ in $\mathbf{R}^n - Q$ ” left the lemma false as stated, thus leaving several papers [2], [3], which quoted [1], in doubt. The present result substantiates the claims in [2], [3]. A similar but less general result has been shown by M. Giaquinta in a paper yet to appear.

It is convenient to first prove a Stieltjes integral lemma. Here it is assumed that $h: [1/k, \infty) \rightarrow [0, \infty)$ is nonincreasing, right continuous and $h(t) \rightarrow 0$ as $t \rightarrow \infty$. Also $H: [1/k, \infty) \rightarrow [0, \infty)$ is measurable, $q > 1$, $a > 1$, $k > 1$ and p satisfies $1 > ak^{p-1}(p - q)/(p - 1)$, with $p \geq q$.

Lemma. *If $-\int_{(t, \infty)} s^{q-1} dh(s) \leq at^{q-1}(h(t/k) + H(t/k))$ for $t \geq 1$ then*

$$-\int_{(1, \infty)} s^{p-1} dh(s) \leq c_1 \left(-\int_{(1, \infty)} s^{q-1} dh(s) \right) + c_2 \int_{(1, \infty)} t^{p-2} H(t/k) dt + c_3 h(1/k).$$

Proof. Let $I'_p = -\int_{(1, J]} s^{p-1} dh(s)$. So by integration by parts we have

$$(1) \quad I'_p = -\int_{(1, J]} t^{p-q} t^{q-1} dh(t) = I'_q + (p - q)J$$

where $J = \int_{(1, J]} t^{p-q-1} \left(-\int_{(t, J]} s^{q-1} dh(s) \right) dt.$

Combining this with the hypothesis we get

$$(2) \quad J \leq \int_{(1, J]} t^{p-q-1} \left(at^{q-1} [h(t/k) + H(t/k)] + \int_{(J, \infty)} s^{q-1} dh(s) \right) dt$$

but $\int_{(1, J]} t^{p-2} h(t/k) dt = k^{p-1} \int_{(1/k, J/k)} t^{p-2} h(t) dt$