

Existence of a Minimal Solution and a Maximal Solution of Nonlinear Elliptic Boundary Value Problems

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1. Introduction. Let N be a natural number and let \mathbf{R} be the set of real numbers. In the following it is assumed that Ω is a nonempty bounded domain of \mathbf{R}^N . We denote by $\bar{\Omega}$ the closed hull of Ω . $\partial\Omega$ means the boundary of Ω . We assume that f maps $\bar{\Omega} \times \mathbf{R}$ into \mathbf{R} . We suppose that g maps $\partial\Omega \times \mathbf{R}$ or sometimes only $\partial\Omega$ into \mathbf{R} . Let L be a uniformly elliptic second order differential operator and let B be a first order boundary operator. Then we consider nonlinear elliptic boundary value problems (BVP's) of the form

$$(1) \quad \begin{aligned} Lu &= F(u) \quad \text{in } \Omega, \\ Bu &= G(u) \quad \text{on } \partial\Omega, \end{aligned}$$

where F is defined by $F(u)(x) := f(x, u(x))$ for every $u: \bar{\Omega} \rightarrow \mathbf{R}$ and every $x \in \bar{\Omega}$. G is defined by $G(u)(x) := g(x, u(x))$ for every $u: \bar{\Omega} \rightarrow \mathbf{R}$, $x \in \partial\Omega$ respectively $G(u)(x) := g(x)$ for every $x \in \partial\Omega$. For precise definitions of L , B , f and g we refer to the next paragraph. By a solution of problem (1) we mean a classical solution.

There are many articles which guarantee a minimal or maximal solution of BVP's (1) in a fixed domain (e.g. an interval) of a function space (e.g. [1], [2], [3] Theorem 4.1 and 4.2). But no paper could be found which answers the question whether a minimal or maximal solution exists with respect to all solutions. In this paper we provide a positive answer for certain BVP's of the form (1). As a simple consequence we get two added solutions if we know only two solutions which are not comparable. A main advantage is based on the following fact. If we approximate the minimal solution and the maximal solution ([4], [5]), then automatically we get a domain which contains every solution of the problem. This may be important in technical problems.

Definitions. A nonempty closed subset P of a real Banach space Z satisfying $P + P \subset P$, $\alpha P \subset P$ for all $\alpha \in \mathbf{R}$ with $\alpha \geq 0$, $P \cap (-P) = \{0\}$ is called a *cone*. Every cone P in Z defines an ordering in Z by $x \leq y$ if