

Two Applications of a Nonlinear Integral Formula to Analytic Functions

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Let $\Omega \subset \subset \mathbb{C}^n$ be an open set with a smooth defining function $r \in C^2(\bar{\Omega})$, $\Omega = \{r < 0\}$, $dr \neq 0$ on $\partial\Omega = \{r = 0\}$ such that r is strongly plurisubharmonic at each point of $\bar{\Omega}$. Let $\beta = (i/2) \sum dz_j \wedge d\bar{z}_j$ be the standard Kähler form on \mathbb{C}^n and $d^c = i(\bar{\partial} - \partial)$. The following formula for $f = u + iv \in C^2(\Omega) \cap C^1(\bar{\Omega})$ was introduced in [2, page 378]:

$$(1) \quad \int_{\Omega} r [(dd^c u)^2 + (dd^c v)^2] \wedge \beta^{n-2} = - \int_{\Omega} |\bar{\partial} f|_r^2 + \int_{\partial\Omega} |\bar{\partial}_b f|_r^2$$

where

$$|\bar{\partial}_b f|_r^2 = d^c r \wedge 2i_b \bar{f} \wedge \bar{\partial}_b f \wedge \beta^{n-2}$$

is nonnegative on $\partial\Omega$ because $\beta > 0$, and

$$|\bar{\partial} f|_r^2 = dd^c r \wedge 2i \bar{\partial} f \wedge \bar{\partial} f \wedge \beta^{n-2}$$

is nonnegative on Ω because $\beta > 0$ and $dd^c r > 0$.

First we note that this gives a new proof of a theorem of Bochner [3]. A related argument has also been used by Siu [8].

Theorem 1. *Let $f \in C^1(\bar{\Omega})$ be harmonic on Ω . Then*

$$\int_{\partial\Omega} |\bar{\partial}_b f|_r^2 \geq \int_{\Omega} |\bar{\partial} f|_r^2$$

and, in particular, f is holomorphic on Ω if and only if $\bar{\partial}_b f$ vanishes on $\partial\Omega$.

Proof. From the formula (1), we only have to show that $(dd^c u)^2 \wedge \beta^{n-2}$ and $(dd^c v)^2 \wedge \beta^{n-2}$ are nonpositive; i.e. nonpositive multiples of the volume form $(1/n!) \beta^n$. At a given point of Ω , a unitary change of coordinates can be made so that $dd^c u = (i/2) \sum \lambda_j dz_j \wedge d\bar{z}_j$. Then $\sum \lambda_j = \Delta u =$