

Upper Semi-continuity and the Monge-Ampère Differential Inequality

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1. Introduction. We shall operate in Euclidean n -space, \mathbf{R}^n , $n \geq 2$, and set

$$(1.1) \quad \mathcal{L}u = (-1)^n \det(\partial^2 u / \partial x_i \partial x_j).$$

In this paper, we shall refer to

$$(1.2) \quad \mathcal{L}u \leq 0$$

as the Monge-Ampère differential inequality (see [2, page 326] and [4, page 152]). In particular, for $n = 2$,

$$\mathcal{L}u = \partial^2 u / \partial x_1^2 \partial^2 u / \partial x_2^2 - (\partial^2 u / \partial x_1 \partial x_2)^2.$$

$B(x^\circ, r_\circ)$ will designate the open n -ball with center x° and radius r_\circ . If u is in $L^1[B(x^\circ, r_\circ)]$ and if there is a polynomial $P(x)$ without constant term of degree k , k a positive integer, such that

$$(1.3) \quad r^{-n} \int_{B(0,r)} |u(x^\circ + x) - u(x^\circ) - P(x)| dx = o(r^k)$$

as $r \rightarrow 0$, u will be said to be in $t_k^1(x^\circ)$ (see [1, page 172]). If $k = 1$, we set $D_{ij}u(x^\circ) = \partial P(x) / \partial x_j$. If $k = 2$, we set $D_{ij}u(x^\circ) = \partial^2 P(x) / \partial x_i \partial x_j$. Clearly, if u is in $t_2^1(x^\circ)$, then u is also in $t_1^1(x^\circ)$.

We shall say $\mathcal{L}u \leq 0$ pointwise at x° provided u is in $t_2^1(x^\circ)$ and furthermore $(-1)^n \det[D_{ij}u(x^\circ)] \leq 0$. In particular, it is clear that if u is in C^2 in a neighborhood of x° and satisfies (1.2) classically at x° , then u satisfies (1.2) pointwise at x° .

Next, let Ω be a fixed bounded open connected set in \mathbf{R}^n containing the origin which is starlike with respect to the origin. We shall suppose also that there is a function g continuous in \mathbf{R}^n with the following seven properties:

$$(1.4) \quad \partial\Omega = \{x : g(x) = 1\} \quad \text{and} \quad \Omega = \{x : g(x) < 1\}.$$

$$(1.5) \quad g(0) = 0.$$

$$(1.6) \quad g(t_1 x) < g(t_2 x) \quad \text{for} \quad 0 \leq t_1 < t_2 \leq 1 \quad \text{and} \quad x \text{ in } \partial\Omega.$$