

On the Marcinkiewicz Integral and Functions of Bounded Mean Oscillation

ALBERTO TORCHINSKY & RICHARD L. WHEEDEN

Introduction. A result due to M. Weiss and A. Zygmund [11] states that if $F(x)$ is periodic and for some $\beta > 1/2$ satisfies the smoothness condition

$$\Delta^2 F(x;t) = F(x+t) + F(x-t) - 2F(x) = O(t|\log t|^{-\beta}),$$

$t > 0$, uniformly in x , then F is the indefinite integral of a function belonging to L^p for every $p < \infty$. An example showing that the conclusion fails when $\beta = 1/2$ is also given. The result for $\beta > 1/2$ was strengthened by John and Nirenberg [4], who showed by a different technique that F' is actually of bounded mean oscillation. One purpose of this note is to obtain essentially the same conclusion under weaker assumptions, and to show there are n -dimensional and local versions of the result.

The smoothness condition above of course becomes stronger as β increases. In case $\beta = 0$, it is the classical Λ^* condition of Zygmund, and is satisfied by the indefinite integral of a function of bounded mean oscillation. The following particularly simple proof of this fact is due to A. P. Calderón. Let $\theta(s)$ be the odd function equal to 0 for $|s| \geq 1$, -1 for $-1 < s < 0$, and $+1$ for $0 < s < 1$. Then if $F(x) = \int^x f$ and c is any number,

$$\begin{aligned} |\Delta^2 F(x;t)| &= \left| \int f(s) \theta\left(\frac{x-s}{t}\right) ds \right| \\ &= \left| \int [f(s) - c] \theta\left(\frac{x-s}{t}\right) ds \right| \\ &\leq \int_{|x-s| < t} |f(s) - c| ds. \end{aligned}$$

Choosing c to be the integral average of f over $\{s: |x-s| < t\}$, we see that the last expression is $O(t)$ if f is of bounded mean oscillation (see the definition which follows). We give below another condition which is necessary and, in a sense, also sufficient for F to be the integral of a function of bounded mean oscillation.