

Boundary Regularity for Variational Inequalities

ROBERT JENSEN

1. Introduction. We present a boundary regularity result which in some sense generalizes the work of Brezis and Kinderlehrer [1]. To be precise, we prove boundary regularity if the operator in the variational inequality is a linear elliptic operator. However, our methods may easily be adapted to prove the interior regularity result in [1] for a monotone vector field.

We shall now proceed to a more detailed explanation of our work. The discussion of our results is facilitated by introducing the following definitions and notation.

(1.1) Ω , an open domain in \mathbf{R}^n with smooth boundary, $\partial\Omega$.

$$L \equiv \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + c(x),$$

a symmetric uniformly elliptic operator with $a_{ij}(x) \in C^{2,\alpha}(\bar{\Omega})$, $b_i(x) \in C^\alpha(\bar{\Omega})$ for $1 \leq i, j \leq n$, $c(x) \in C^\alpha(\bar{\Omega})$ and $c(x) \geq 0$.

(1.3) ϕ , a function in $C^{1,\alpha}(\bar{\Omega})$ (the obstacle).

(1.4) g , a function in $C^\alpha(\partial\Omega)$ and $g \geq \phi|_{\partial\Omega}$ (the boundary data).

(1.5) f , a function in $L^2(\Omega)$.

(1.6) $\mathbf{K} \equiv \{v \in H^1(\Omega) : v|_{\partial\Omega} = g \text{ and } v \geq \phi\}$.

(1.7) $\langle \cdot, \cdot \rangle$ the L^2 inner product or the usual evaluation map on $H^{-1} \times H_0^1$.

(1.8) w , the solution of the variational inequality: $w \in \mathbf{K}$ and for all $v \in \mathbf{K}$ $\langle Lw, v - w \rangle \geq \langle f, v - w \rangle$.

We shall now discuss the two regularity results. Let x_0 be a point in Ω . From [1] it follows that if there is a neighborhood, N_1 , of x_0 such that $f \in C^\alpha(N_1 \cap \Omega)$ and $\phi \in C^{1,1}(N_1 \cap \Omega)$ then there is a neighborhood, N_2 , of x_0 such that $w \in W^{2,\infty}(N_2)$. By interpreting the second derivatives of the solution as functions satisfying Neumann conditions at the boundary instead of just Dirichlet conditions, we have been able to extend the previous