

Weak Estimates for the Parabolic Hardy-Littlewood Maximal Function Through Unbounded Sets in R^m

FELIPE ZO & HUGO ALVAREZ

1. Introduction and results. This note is concerned with the problem of extending the Lebesgue differentiation theorem of the integral in the euclidean space R^m , when the differentiation is carried out through a suitable family of unbounded sets. The first result in this direction appeared in [1], where a homothetic family to a star-shaped set is considered. We can also call any of these sets an isotropic dilation of a star-like set. Our interest is to replace isotropic dilations by non-isotropic ones.

First some notations. For an x in R^m and $\lambda > 0$ we denote by $\lambda \cdot x$ the λ non-isotropic dilation of x . More precisely, following the notation in [3], $\lambda \cdot x = T_\lambda x = \exp(P \log \lambda)(x)$, where P is an $m \times m$ real positive definite matrix, the infinitesimal generator of the group $\{T_\lambda\}_{\lambda > 0}$. As is well known, $\det T_\lambda = \lambda^a$, $a = \text{trace } P$. The euclidean norm of x in R^m will be denoted by $|x|$, and $\|x\|$ will stand for the metric associated to the family of non-isotropic dilations. It is known that $\|\lambda \cdot x\| = \lambda \|x\|$. More details can be found in [3].

Throughout this note E will be a measurable set in R^m which is star-like with respect to $\{T_\lambda\}$, that is $E_\lambda = T_\lambda E \subseteq E$ if $0 < \lambda \leq 1$. Moreover E will have finite measure. The problem concerning us will be the existence of the limit

$$(1.1) \quad \lim_{\lambda \rightarrow 0} \frac{1}{|E_\lambda|} \int_{E_\lambda} f(x+y) dy \quad \text{a.e.,}$$

for $f \in L^p(R^m)$, $1 \leq p \leq \infty$. As is usual we shall estimate the maximal function $f^*(x) = \sup_{\lambda > 0} (1/|E_\lambda|) \int_{E_\lambda} |f(x+y)| dy$. Since in our context there is no result similar to Lemma 2.1 in [1], it is not possible to find estimates for f^* using only the argument given there.

It is easy to obtain $\|f^*\|_p \leq C_p \|f\|_p$ for $p > 1$. In fact, if $\phi(x)$, $|x| = 1$, denotes the boundary function of the set E , that is if $\phi(x) = \sup \{\lambda > 0: \lambda \cdot x \in E\}$, we have: $x \in E$ iff $\|x\| \leq \phi(x')$, where $x' = \|x\|^{-1} \cdot x$. Now, using polar-like coordinates, see [3], we have