

The Motion of the Free n -dimensional Rigid Body

TUDOR RATIU

§1. Introduction. This paper deals with the complete integrability of the equations of motion of a free n -dimensional rigid body about a fixed point. The Euler vector fields on a Lie algebra and its dual are reviewed in §2. Euler's equations of motion for the n -dimensional free rigid body about a fixed point as a Hamiltonian system on adjoint orbits of $so(n)$ are derived in §3 and Manakov's constants of the motion are given. Inspired by the proof of the Kostant-Symes involution theorem, §4 gives a Lie algebraic proof of the involution of Manakov's integrals. Based on work of Kupershmidt, Manin, Adler and van Moerbeke, a Hamiltonian structure different from the one defined by the decomposition of $so(n)$ in its adjoint orbits is given in §5 and Lenard type recursion relations between the two families of Hamiltonian vector fields are found. This enables us to give a second proof—entirely similar to the one for the Korteweg-de Vries equation—of the involution of Manakov's integrals. §6 deals with another set of integrals of the Euler equations found earlier by Mishchenko. Finally, §7 extends a matrix equation of Dubrovin to a Hamiltonian system on adjoint orbits of $s\ell(n)$ which, when restricted to $so(n)$, gives the rigid body equations; Manakov and Mishchenko type integrals are shown to be in involution.

When this present work was completed, B. Kupershmidt and V. Kac mentioned the announcement [17] and the paper [18] respectively of which we were not aware. There, Mishchenko and Fomenko generalize the rigid body equations to any semi-simple Lie algebra in the spirit of our §7 and prove complete integrability. Our proofs of involution are however different but the independence in which we used crucially the Lenard relations, turns out to be very similar to their proof; we refer the reader to [18, Section 4] for this proof. For an algebraic geometrical proof of independence and the linearization of the flow, see Adler-van Moerbeke [3].

I want to express my gratitude to P. van Moerbeke for his encouragement and support during the work on this paper. Many thanks to J. Marsden for discussions, a critical reading of the manuscript and suggestions. Conversations with M. Adler, B. Kupershmidt, R. Palais, G. Schwartz and J. Wolf as well as the referee's comments and bibliographical additions ([6], [7], [23], [2])