

Orthogonal Complements to Invariant Subspaces

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Introduction. Let D be a finitely connected domain in the plane bounded by $n + 1$ analytic Jordan curves. That is, we have $\partial D = \gamma_1 \cup \dots \cup \gamma_{n+1}$. Let $H^2(D)$ denote the usual class of holomorphic functions f on D , i.e. $|f|^2$ has a harmonic majorant. $H^\infty(D)$ is the space of all holomorphic functions on D which are bounded. A closed subspace M of H^2 is said to be invariant if $fg \in M$ for all $f \in M$ and $g \in H^\infty$. The theory of such subspaces is well known. For $D = U$, the unit disk, it can be found in Hoffman [5]. For the general case, see Voichick [7], or Sarason [6]. Each M has the form $M = sH^2$, where $s \in H^\infty$ is an inner function. That is, $|s(z)| \leq 1$ on D , and

$$|s| = 1 \text{ a.e. on } \gamma_{n+1} \quad \text{and}$$

$$|s| = c_i \text{ a.e. on } \gamma_i \quad i = 1, \dots, n.$$

“Almost everywhere” is with respect to ds , arclength.

In [1], Ahern and Clark study $M^\perp = H^2(U) \ominus sH^2(U)$. An important special case occurs when s has the form

$$s(z) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta)\right)$$

where $d\sigma > 0$ is nonatomic and singular with respect to $d\theta$. They construct an operator $V: L^2(d\sigma) \rightarrow M^\perp$ given by

$$(Vc)(z) = \int_0^{2\pi} \sqrt{2} c(\lambda) s_\lambda(z) (1 - e^{-i\lambda} z)^{-1} d\sigma(\theta),$$

where $s_\lambda(z) = \exp(-\int_0^\lambda e^{i\theta} + z/e^{i\theta} - z d\sigma(\theta))$. Ahern and Clark prove that V is an isometry of $L^2(d\sigma)$ onto M^\perp . As an application they study the operator V^*TV on L^2 , where T is the restricted shift on M^\perp : $Tf = Pfz$, where P is orthogonal projection onto M^\perp . It turns out that V^*TV is the sum of multiplication by z and a compact integral operator.

It is the purpose of this paper to generalize the results of Ahern and Clark to $M^\perp = H^2(D) \ominus sH^2(D)$. Although we can handle an arbitrary inner function,