

On Nemytskii's Operator and its Application to the Lower Semicontinuity of Integral Functionals

R. LUCCHETTI & F. PATRONE

1. Introduction. This work is devoted to the study of some properties of Nemytskii's operator, with applications to the semicontinuity of an integral functional. Given $I \subset \mathbf{R}^n$ and a Carathéodory function $f: I \times \mathbf{R} \rightarrow \mathbf{R}$, Nemytskii's operator is defined by $(Nx)(t) = f(t, x(t))$ $t \in I$. It is well known that a measurable function x is mapped by N into a measurable function Nx . A classical result states that if the operator N is defined on $L^p(I)$ with values in $L^q(I)$, then it is bounded and continuous (see [8], [13]). Moreover, N maps all of $L^p(I)$ into $L^q(I)$ if and only if the function f satisfies the following condition:

$$(*) \quad \begin{aligned} |f(t, x)| &\leq a(t) + b|x|^{p/q} && \text{with } a \in L^q(I), b > 0 \text{ if } q < +\infty \\ |f(t, x)| &\leq M && \text{if } q = +\infty. \end{aligned}$$

We consider Nemytskii's operator N , when $f: I \times X \rightarrow Y$. Here X and Y are separable real Banach spaces, and (I, S, m) is a given measure space. We get the following results.

1) The boundedness of $N: L^p(I, X) \rightarrow L^q(I, Y)$ follows directly from the fact that N is actually defined between these spaces, without any regularity assumption of f .

2) If f is $S \times B(X)$ measurable (see Section 2 for notations and basic assumptions) then N maps all of $L^p(I, X)$ into $L^q(I, Y)$ if and only if (*) holds.

3) If f is a Carathéodory function, then $N: L^p(I, X) \rightarrow L^q(I, Y)$ is continuous. By means of recent results by Ioffe, we prove a partial converse of the last point. Actually, if $f: I \times \mathbf{R}^n \rightarrow \mathbf{R}^m$ is supposed to be $S \times B(\mathbf{R}^n)$ measurable then we prove that the Carathéodory condition for f is necessary for the continuity of $N: L^p(I, \mathbf{R}^n) \rightarrow L^q(I, \mathbf{R}^m)$.

About the need for the above generalizations of the classical properties of Nemytskii's operator, we wish to point out the study of optimal control problems for differential equations in infinite dimensional spaces. Such problems arise, for example, from the study of the optimal control theory