

# *Full Houses and Cones of Excessive Functions*

THOMAS E. ARMSTRONG

**0) Introduction.** A gambler in a gambling house  $\Gamma$  may be faced with a lower bounded payoff function  $s$  which is such that no matter which fortune or distribution of fortunes he possesses he is always better off not taking a gamble available other than staying put at his initial fortune (if this is available). Such payoff functions are called excessive functions. Excessive functions form a convex cone which contains the constants and is closed under arbitrary lower bounded pointwise infima. We show that any cone of lower bounded functions on a set  $F$  which has these properties is the cone of excessive functions for some gambling house  $\Gamma$  on  $F$ , Proposition 5.

If  $\mathcal{E}$  is a cone of excessive functions for a gambling house  $\Gamma$  on the fortune space  $F$  there may be other houses with the same cone of excessive functions. We show that there is a largest such house, the full house  $\hat{\Gamma}$ , associated with  $\Gamma$ . The gambles  $\gamma$  in  $\hat{\Gamma}$  available at a fortune  $f$  are precisely those such that no matter which excessive payoff function  $s$  the gambler faces he would be better off staying at  $f$  than taking the gamble  $\gamma$ . That is,  $\int s d\gamma \leq s(f)$  for all  $s \in \mathcal{E}$ . This is the definition of the relation  $\delta_f < \gamma$ . More generally, if a gambler has initial distribution of fortunes  $\nu$  and  $\gamma$  is another distribution such that the expected payoff under  $\nu$  for any excessive payoff function  $s$  is greater than the expected payoff under  $\gamma$ ,  $\gamma$  is said to dominate  $\nu$  and we write  $\gamma > \nu$ . We show how to construct the full house  $\hat{\Gamma}$  from  $\Gamma$ . In fact we show that  $\gamma > \nu$  iff there is a finitely additive probability distribution  $p$  on the set of policies available in  $\Gamma$  with initial distribution  $\nu$  so that  $\gamma$  is the  $p$  average of the terminal distributions of these policies. Thus,  $\gamma$  is a randomized policy distribution starting from the initial distribution  $\nu$ . The full house  $\hat{\Gamma}$  has available at a fortune  $f$  the randomized terminal distributions of policies available in  $\Gamma$  at  $f$ .

We conclude with several examples of cones which are cones of excessive functions for some gambling house. Concave functions on a convex set, monotone functions on a partially ordered set, hypermedian functions in potential theory (both axiomatic and probabilistic) all form excessive cones.