

Automorphisms of C^ -algebras and Second Čech Cohomology*

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Let A be a C^* -algebra with identity, and let $\text{Aut } A$ be the group of all $*$ -automorphisms of A endowed with the norm topology. In their systematic investigation [22] of $\text{Aut } A$, Kadison and Ringrose considered the subgroups $\text{Inn } A$, $\gamma(A)$, $\pi(A)$ of $\text{Aut } A$ consisting of, respectively, inner automorphisms, those path-connected to the identity, and π -inner automorphisms. As well as proving some general theorems about the relationships between these subgroups, they investigated in detail some particular cases, including the algebra $C(X, M_n(\mathbb{C}))$ of continuous functions from a compact space into the matrix algebra $M_n(\mathbb{C})$. Subsequently Lance [25] and Smith [32] considered the algebra $A = C(X, B(H))$, where X is compact and separable, and $B(H)$ is the algebra of all operators on a Hilbert space H of dimension \aleph_0 . They proved the striking result that $\pi(A)/\gamma(A) \cong H^2(X, \mathbb{Z})$, the second Čech cohomology group of X with integral coefficients. It follows from their results that in this case $\pi(A)$ coincides with the group $\text{Aut}_{C(X)} A$ of automorphisms which preserve the $C(X)$ -module structure of A , and that $\gamma(A) = \text{Inn } A$; hence their theorem identifies the group $\text{Aut}_{C(X)} A / \text{Inn } A$ of outer $C(X)$ -automorphisms of $A = C(X, B(H))$ with $H^2(X, \mathbb{Z})$.

We shall extend the theorem of Lance and Smith to C^* -algebras of sections of bundles whose fibres are C^* -algebras. We consider two distinct types of bundles: the first have as fibre the algebra $K(H)$ of compact operators on a Hilbert space H , and as structure group $\text{Aut } K(H)$ equipped with the topology of pointwise convergence; the algebra $A = \Gamma_0(E)$ of sections of such a bundle E over a locally compact space X is called a stable continuous trace C^* -algebra with spectrum X . We prove that if $\text{Inn } A$ now denotes the group of automorphisms which are implemented by multipliers (this is consistent notation if A has identity), then there is a short exact sequence

$$0 \rightarrow \text{Inn } A \rightarrow \text{Aut}_{C_b(X)} A \xrightarrow{\pi} H^2(X, \mathbb{Z}) \rightarrow 0.$$

In fact, if A is any separable continuous trace C^* -algebra then we can construct the homomorphism π , but it is not necessarily surjective. The second type of bundle we consider has as fibre a C^* -algebra B with identity and structure group $\text{Inn } B$ in its norm topology. If E is such a bundle over a