

The Distortion Coefficient of an Operator with Bounded Characteristic Function

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1. Introduction. Let T be an operator on a Hilbert space \mathfrak{H} , and suppose that T is similar to a contraction: $\|STS^{-1}\| \leq 1$ for some invertible operator S from \mathfrak{H} to a Hilbert space \mathfrak{H}' . The *distortion coefficient* of T is the number $M(T)$ defined by

$$(1.1) \quad M(T) = \inf \{ \|S\| \cdot \|S^{-1}\| : \|STS^{-1}\| \leq 1 \}.$$

$M(T)$ can also be defined by

$$(1.2) \quad M(T) = \inf \{ m^{-1}M : |T| \leq 1 \text{ with respect to some inner product norm } |\cdot| \text{ satisfying } 0 \leq m\|h\| \leq |h| \leq M\|h\| \text{ for all } h \in \mathfrak{H} \}.$$

(See [5] and [6].)

A sufficient condition for T to be similar to a contraction is that its characteristic function Θ_T be bounded:

$$(1.3) \quad \sup \{ \|\Theta_T(\lambda)\| : |\lambda| < 1 \} = C < \infty.$$

(See [3] for a definition of the characteristic function and a proof of this result.) We obtain here an estimate for $M(T)$ in terms of C :

Theorem 1. *If C is given by (1.3) and $K = (2C^2 + 1)^{1/2}$, then $M(T) \leq \sqrt{2} K^3 (K^2 + 1)^{1/2}$.*

In proving this theorem we make use of a dilation U of T , constructed by Ch. Davis [2], in which the Hilbert space \mathfrak{H} is considered as a subspace of a larger space \mathfrak{R} . On \mathfrak{R} we have an indefinite inner product $[\cdot, \cdot]$ (i.e., $[h, h]$ can be negative for some $h \in \mathfrak{R}$) and a Hilbert space inner product (\cdot, \cdot) . The two inner products are related by

$$(1.4) \quad (h, k) = [Jh, k],$$

where J is a symmetry on \mathfrak{R} : $J^2 = I$ and $J^* = J$. (Throughout this paper, A^* will denote the adjoint of an operator A formed with respect to the Hilbert space inner product: $(Ah, k) = (h, A^*k)$. With J^* , however, it clearly makes no difference (because of (1.4)) which inner product is used.) We let $\|\cdot\|$ denote the J -norm on \mathfrak{R} :

$$\|h\| = (h, h)^{1/2} = [Jh, h]^{1/2}.$$

We call any space with this structure a *Krein space* (see [1] and [8]).