

Lower Bounds for Eigenvalues of Sturm Liouville Systems

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I. Introduction. Subject to mild assumptions on the coefficients, it is well known [4] that the equation

$$(1) \quad [P(t)y']' + [\lambda\rho(t) - Q(t)]y = 0 \quad y(0) = y(a) = 0$$

has an infinite sequence of eigenvalues $\lambda_n = \lambda_n(P, Q, \rho)$ which depend on the coefficient functions. The eigenvalues are ordered so that

$$\lambda_1(P, Q, \rho) < \lambda_2(P, Q, \rho) < \lambda_3(P, Q, \rho) < \dots$$

Using the well known Liouville transformation [4; page 292] it follows that any equation of form (1) may be transformed into one of the form:

$$(2) \quad y'' + (\lambda - q(x))y = 0 \quad y(0) = y(l) = 0.$$

Equations (1) and (2) have the same eigenvalues λ_n but to indicate whether they are calculated using (1) or (2) we use the notation $\lambda_n = \lambda_n(P, Q, \rho)$ or $\lambda_n = \lambda_n(q)$. A great deal of attention has been devoted to $\lambda_n(P, Q, \rho)$. We quote one result of the type we have in mind which is due to Krein [7];

Theorem. If $0 \leq \rho(x) \leq H$ and $\int_0^a \rho(x)dx = m$ then

$$\frac{4Hn^2}{m^2} X\left(\frac{m}{aH}\right) \leq \lambda_n(1, 0, \rho) \leq \frac{n^2\pi^2 H}{m^2}.$$

Here $X(t)$ denotes the smallest positive root of the equation $X^{1/2} \tan X^{1/2} = t(1-t)^{-1}$. Krein's result applies to equations of the form $y'' + \lambda\rho(x)y = 0$. This is similar to our Theorem 2 below. Theorem 2 applies however to equations of the form $y'' + (\lambda - q(x))y = 0$ and thus indirectly to the most general equation (1).

For other results concerning λ_n see [1], [2], [3], [7], [9] and the references quoted there. We state another result due to B. Schwarz [9]:

$$\lambda_n(1, 0, \bar{\rho}_{-n}) \leq \lambda_n(1, 0, \rho) \leq \lambda_n(1, 0, \bar{\rho}_{+n}).$$

Here $\bar{\rho}_{+n}$ denotes the rearrangement of the function ρ into symmetrically increasing order and $\bar{\rho}_{-n}$ is its rearrangement into symmetrically decreasing