

On the Construction of the Reduced Phase Space of a Hamiltonian System with Symmetry

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Introduction. Given a principal G -bundle $\pi:M \rightarrow N$, $A.$ Weinstein in [1] associates with each connection ω on the bundle a lift $\tilde{\pi}_\omega:T^*M \rightarrow T^*N$ of π to the cotangent bundles which is constant along G -orbits of the exact symplectic lifted group action. He uses this correspondence between connections ω and lifts $\tilde{\pi}_\omega$ in his construction of Sternberg's phase space for a particle in a Yang-Mills field. (See e.g. [2], [3], [7].) In the present paper the same correspondence is used in the construction of the reduced phase space of a Hamiltonian system with symmetry group G . Previous work on the reduced phase space can be found e.g. in [5]–[11].

Let $\pi:M \rightarrow N$, ω and $\tilde{\pi}_\omega$ have the same meanings as before, let $\psi:T^*M \rightarrow \mathcal{G}^*$ (\mathcal{G}^* = dual of Lie algebra of G) be the moment map of the lifted group action and let $\mu \in \mathcal{G}^*$ be G -invariant. We shall demonstrate that $\tilde{\pi}_\omega$ induces a symplectomorphism between the reduced phase space $\psi^{-1}(\mu)/G$ and T^*N . Here, the 2-form $\sigma_{\omega,\mu}$ defining the symplectic structure on T^*N differs from $d\Theta_N$ (Θ_N = canonical 1-form) by an additive "magnetic term" which turns out to be the μ -component Ω_μ of the curvature of ω , viewed as a 2-form on T^*N (Theorem 1). In particular we shall show that this term can be transformed away by a judicious choice of the connection ω if and only if the G -bundle $\pi:M \rightarrow N$ admits a connection such that Ω_μ is exact (Theorem 2). Finally, we generalize the result of Theorem 1 to the case in which G is a subgroup of a larger group \hat{G} whose lifted action has moment map $\hat{\psi}$. In this situation we give sufficient conditions under which the reduced phase space $\hat{\psi}^{-1}(\mu)/G$ ($\mu \in \hat{\mathcal{G}}^*$) is realized as a symplectic submanifold of T^*N , where T^*N is endowed with an appropriate symplectic structure (Theorem 3).

Incidentally, a comparison of our result with a corresponding result of Marsden and Abraham ([5, page 300]) shows that ours contains a specific recipe for the construction of their G -invariant 1-form α_μ on M which is valid under different assumptions from theirs given on page 345–47. Under our assumptions this recipe is expressed by the formula $\alpha_\mu(p) = \omega_p^*\mu$, ω_p^* being the dual of