

When Two Entire Functions and also their First Derivatives Have the Same Zeros

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1. Introduction. We say two entire functions $f(z)$ and $g(z)$ share the value a CM when $f(z) - a$ and $g(z) - a$ have the same zeros counting multiplicities. C. C. Yang has proven:

Theorem A. [7] Suppose two transcendental entire functions f and g satisfy the following three conditions:

- (a) f and g share 0 CM, and all the zeros are simple.
- (b) f' and g' share 0 CM.

(c) $\rho = \max \left[\overline{\lim}_{r \rightarrow \infty} \log \log \log M(r, f) / \log r, \right.$

$\left. \overline{\lim}_{r \rightarrow \infty} \log \log \log M(r, g) / \log r \right] < 1.$

Then f and g satisfy exactly one of the following two relations:

- (I) $f(z) = c(g(z))^k$ where c and k are constants, or
- (II) $f(z) = c_1 e^{\gamma(z)} + c_2$, $g(z) = c_3(c_2 e^{-\gamma(z)} + c_1)$, where c_1 , c_2 , and c_3 are constants and γ is entire of order less than one.

His example $f(z) = \exp(e^z)$ and $g(z) = e^{2z}$ shows that condition (c) is necessary in Theorem A. He also notes that the conclusion still holds if (a) is replaced by (a') f and g share 0 CM and the exponent of convergence of the multiple zeros is less than $1 - \rho$.

We will prove:

Theorem 1. f and g are entire functions of finite order such that f , g share 0 CM and f' , g' share 0 CM if and only if we have exactly one of the following four cases:

- 1. $f(z) = Cg(z)$ where $C \neq 0$ is a constant and f is entire with order $(f) < \infty$;
- 2. $f(z) = e^{p(z)}$, $g(z) = ae^{bp(z)}$, where $a \neq 0$, and $b \neq 0, 1$, are constants, and p is a non-constant polynomial;
- 3. $f(z) = a(e^{p(z)} - 1)^n$, $g(z) = b(1 - e^{-p(z)})^n$, where a , b are non-zero constants, n is a positive integer, and p is a non-constant polynomial;

4. $f(z) = \exp \left[\sum_{n=0}^{2N} a_n (2\pi i)^{-n} \int \frac{p'(z)(p(z))^n}{1 - e^{-p(z)}} dz \right],$