

On Interpolation of Martingale L^p Spaces

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1. Introduction. In [9] Peetre proposed the problem of characterizing the interpolation spaces between martingale L^p spaces. It was conjectured that $(ML^{p_0}, ML^{p_1})_{\theta, q} = ML(p_\theta, q)$, $1 \leq p_0 \neq p_1 \leq \infty$, $0 < q \leq \infty$, $0 < \theta < 1$, $1/p_\theta = (1 - \theta)/p_0 + \theta/p_1$ (see §2 below for the definition of these spaces).

R. Cignoli [3] obtained a partial result by means of proving that $ML(p, q) \approx L(p, q)$, $1 < p < \infty$, $1 \leq q < \infty$ (see also (2.3) below).

The purpose of this note is to prove the conjecture of Peetre for the full range of variation of the parameters, i.e. we prove the following.

Theorem A. Let $1 \leq p_0 \neq p_1 \leq \infty$, $0 < \theta < 1$, $0 < q \leq \infty$, then

$$(ML^{p_0}, ML^{p_1})_{\theta, q} = M(L^{p_0}, L^{p_1})_{\theta, q} = ML(p_\theta, q)$$

where

$$\frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.$$

Our method is based on the computation of Peetre's K functional for the martingale (ML^p, ML^∞) couples and the use of the reiteration theorem for interpolation spaces (cf. [1]).

Similar ideas can be used to characterize the interpolation spaces between MH^p , $MBMO$, ML^p and will be treated in [8]. For example a consequence of our method of proof of Theorem A is the following (cf. [12], [11]).

Corollary. $(MH^1, ML^\infty)_{\theta, q} = ML(p_\theta, q)$.

We further remark that the above results also hold in the setting of $L(p, q)$ spaces as it readily follows by means of the reiteration theorem.

We shall assume the reader to be familiar with the elements of the K -theory of interpolation and shall adhere to the notation of [1].

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2. Preliminaries. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $\{B_n\}_{n=0}^\infty$ be an increasing and generating sequence of sub- σ -fields of \mathcal{A} . In the sequel we consider martingales $f = (f_1, f_2, \dots)$ relative to $\{B_n\}_{n=0}^\infty$.

Let $X(\Omega, \mathcal{A}) = X$ be a quasi-normed rearrangement invariant space (cf.