

# *Analyticity of the Boundary for Lipschitz Domains without the Pompeiu Property*

STEPHEN A. WILLIAMS

**1. Introduction.** Let  $D$  be a nonempty bounded open subset of  $\mathbf{R}^n$ . In this paper we will always assume that  $n \geq 2$ . Let  $\Sigma$  denote the set of rigid motions of  $\mathbf{R}^n$  onto itself: Each  $\sigma \in \Sigma$  can be thought of as a rotation followed by a translation. We say that  $D$  has the *Pompeiu property* if and only if the only continuous complex-valued function  $f$  defined on  $\mathbf{R}^n$  for which

$$\int_{\sigma(D)} f(x_1, x_2, \dots, x_n) dx = 0 \quad \text{for every } \sigma \in \Sigma$$

is the function  $f \equiv 0$ . Here  $\sigma(D)$  denotes the image of  $D$  under the rigid motion  $\sigma$ . Balls do not have the Pompeiu property. The conjecture of [10, page 185] remains open, that among bounded open sets of  $\mathbf{R}^n$  whose boundary is homeomorphic to the unit sphere of  $\mathbf{R}^n$ , only the balls fail to have the Pompeiu property.

The boundary of any set  $W$  will be denoted by  $\partial W$ . The *outer boundary* of a bounded open  $D \subseteq \mathbf{R}^n$  is that subset of  $\partial D$  which is in the closure of the unbounded (connected) component of  $\mathbf{R}^n \sim \bar{D}$ . It will be denoted by  $\partial^* D$ . The main result of this paper (stated in Theorem 4) is that if a bounded open  $D \subseteq \mathbf{R}^n$  fails to have the Pompeiu property, if a corresponding number  $\alpha$  (described in the next paragraph) is real, and if an  $(n - 1)$ -dimensional portion of its outer boundary is Lipschitz, then that portion is (real) analytic. Theorem 1 states that if  $D$  fails to have the Pompeiu property and  $\partial^* D = \partial D$ , then  $\alpha$  is real. One consequence of these theorems is that if a bounded open set  $D \subseteq \mathbf{R}^n$  has boundary  $\partial D$  homeomorphic to the unit sphere in  $\mathbf{R}^n$ , if  $D$  fails to have the Pompeiu property, and if  $\partial D$  is Lipschitz, then  $\partial D$  is analytic. This is evidence in support of the conjecture described above. Another consequence of these theorems is that if any convex open  $D \subseteq \mathbf{R}^n$  fails to have the Pompeiu property, then  $\partial D$  is analytic.

It is proved in [10] (from a result in [11]) that if  $D$  fails to have the Pompeiu property then there is at least one  $\alpha \in \mathbf{C}$  with  $\alpha \neq 0$  and a distribution  $T$  of compact support on  $\mathbf{R}^n$  such that  $\Delta T + \alpha T = -\chi_D$  in the distribution sense. (Here and for the rest of the paper, for any set  $W$ ,  $\chi_W$  denotes the characteristic function of  $W$ .) It is also proved in [10] that this distribution