

Isometries on Subspaces of L^p

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Section 0. Introduction. In this paper, we determine the structure of isometric linear operators defined on subspaces of L^p , $p > 0$ and not an even integer. The main tool is a generalization of a theorem of Rudin on L^p isometries and equimeasurability. Our structure results imply that these isometries admit natural (and unique) extensions to isometries defined on explicitly described and potentially much larger subspaces.

In his classic monograph [2], Banach states a result characterizing the isometric linear operators on $L^p [0,1]$, $1 \leq p \neq 2$. His result is that U is an isometry of $L^p [0,1]$ onto itself if and only if there is a one-to-one non-singular measurable map ϕ of $[0,1]$ onto itself and a measurable function h with $|h|^p = d(m \circ \phi^{-1})/dm$ a.e. (where m denotes Lebesgue measure) such that

$$(1) \quad Uf(x) = h(x)f(\phi^{-1}(x)) \quad \text{a.e.}$$

Hence, for $p \neq 2$, L^p isometries are essentially given by such maps ϕ .

Lamperti [4] has proved a generalization of this result to the case $0 < p \neq 2$, μ is a σ -finite measure on X , and $U: L^p(X, \mu) \rightarrow L^p(X, \mu)$ is a (not necessarily onto) linear isometry. Here the space $L^p(X, \mu)$ may be taken real or complex. For Lamperti's result, the point map ϕ must be replaced with a set map—a so-called regular set isomorphism (see Definition 2.1).

Note that if the isometry U of (1) maps the constant function 1 onto itself, then $h(x) \equiv 1$ a.e. and ϕ is measure preserving. This forces every pair of vectors (f_1, \dots, f_n) and (Uf_1, \dots, Uf_n) to be equimeasurable, i.e. $m\{(f_1, \dots, f_n) \in B\} = m\{(Uf_1, \dots, Uf_n) \in B\}$ for all Borel sets B in either \mathbf{R}^n or \mathbf{C}^n (according as $L^p [0,1]$ is taken to be a real or complex vector space).

Rudin [7] has proved that this equimeasurability property holds for isometries preserving 1 which are defined on subspaces of complex L^p :

Theorem 0.1. (Rudin [7]). *Let μ and ν be finite measures on X and Y , respectively, and let $f_1, \dots, f_n \in L^p(X, \mu)$ and $g_1, \dots, g_n \in L^p(Y, \nu)$ be complex-valued functions such that*

$$\int_X \left| 1 + \sum \lambda_j f_j \right|^p d\mu = \int_Y \left| 1 + \sum \lambda_j g_j \right|^p d\nu$$

for all choices of complex numbers $\lambda_1, \dots, \lambda_n$. If $0 < p < \infty$ and p is not