

Geodesics in Riemannian Manifolds-with-Boundary

RALPH ALEXANDER & S. ALEXANDER

§1. Introduction. Variational problems with boundary constraints offer questions of great mathematical and physical interest. (See Almgren's book [1] for a general study of this class of problems.) One of the simplest such variational problems is that of shortest paths in Riemannian manifolds-with-boundary. One would hope that a great deal of specific information might be obtained about this problem, although at present very little seems to be known. It is the purpose of this note, first, to lay some essential groundwork for studying the problem, and secondly, to give a simple construction based on convexity which implies the following theorem:

Theorem 1. Let M be a Riemannian C^3 -manifold-with- C^1 -boundary. A) Any shortest path of M is C^1 . B) At any point where it touches the boundary, a shortest path of M possesses an osculating plane which is normal to the boundary.

The construction is a geometric one which is specific to the Riemannian setting; thus it seems to yield more information and require weaker differentiability hypotheses than known constructions for more general classes of problems. Note that no higher differentiability class than C^1 can be obtained for shortest paths, regardless of the differentiability hypotheses on M (for example, consider the shortest paths in the Euclidean plane with the open unit disc removed).

Part A of Theorem 1 and interesting related results have been proved independently by Franz-Erich Wolter by a different method [9]. A brief discussion of some current related work of several authors is included in the final section of this note.

If M is a Euclidean space with an open convex body removed, then it is easy to see that a shortest path joining points of the boundary ∂M must lie in ∂M . Therefore, it is instructive to compare questions about shortest paths in our setting with their analogues in the theory of convex surfaces (see Busemann's text [3, especially pages 87–94]). The existence of one-sided derivatives of a shortest path γ on a convex surface is well known as Libermann's Theorem. Libermann's proof relies on the construction of surfaces through γ which may be "developed" onto the plane so that the image of