

Homology and Morse Theory of Third Configuration Spaces

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0. Introduction. Let $X^{(r)}$ denote the cartesian product of r copies of a space X . The r^{th} *configuration space* $F(X; r)$ of X (in the sense of Fadell and Neuwirth [3]) is defined by

$$F(X; r) = \{(x_1, \dots, x_r) \in X^{(r)} : \text{if } i \neq j \text{ then } x_i \neq x_j\}.$$

We consider embeddings of a compact, smooth manifold X in a Euclidean space. Extending results of Takens and White [6] we show that if the embedding satisfies a certain natural technical condition, then the Morse inequalities can be used to compute a lower bound for the number of equilibrium positions of r equally charged particles on the embedded manifold. The lower bound depends only on the Betti numbers of $F(X; r)$.

Since $F(X; 2)$ is just the complement of the diagonal in $X \times X$ then, for a reasonably nice space X , the homology of $F(X; 2)$ is easy to deduce from the homology of X . In [1], Cohen calculated the cohomology of $F(X; r)$ for X a Euclidean space. Cohen and Taylor [2] extended that result to compute the cohomology of $F(X; r)$ when X is a sphere or a (noncompact) manifold with a Euclidean factor.

In this paper, we prove that the Betti numbers of $F(X; 3)$ can be computed, for X a compact orientable triangulated manifold without boundary, if the cohomology algebra of X (real coefficients) is known. As an application, we compute the Betti numbers of $F(X; 3)$ for X an oriented surface and use those numbers to obtain a lower bound for the number of equilibrium positions of three equally charged particles on a surface.

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1. Morse theory of configuration spaces. Let $f: X^m \rightarrow E^n$ be a C^k ($k \geq 2$) embedding of a connected compact m -dimensional differentiable manifold into Euclidean n -space, $n > m > 1$. Extending the definition in [6], the embedding f is said to be *V-generic* if the function $V_f: F(X; r) \rightarrow R$ defined by

$$V_f(x_1, \dots, x_r) = \sum_{1 \leq i < j \leq r} \frac{1}{\|f(x_i) - f(x_j)\|}$$