

Principal Curvature and Harmonic Analysis

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1. Introduction. Although the Fourier transform on \mathbf{R}^n is defined in terms of the linear (group) structure, results of recent years have demonstrated a remarkable connection between harmonic analysis and a highly nonlinear object: the unit sphere S^{n-1} . Stein (see Fefferman [2]) was the first to notice that the Fourier transform of an L^p function has a well-defined restriction in $L^2(S^{n-1})$ if p is close enough to 1, even though S^{n-1} has measure 0 in \mathbf{R}^n and \hat{f} is, a priori, defined only almost everywhere. The proofs of this and related results all used explicit formulas for the Fourier transform of the surface measure on S^{n-1} , even though arguments in low dimensions made it clear that the crucial factor was the curvature of S^{n-1} (see Tomas [10], Strichartz [9], Fefferman [2] and Zygmund [11]).

In another direction, Stein and Wainger [7] showed that the maximal operator associated with averages over dilates of S^{n-1} is bounded on L^p , $p > n/(n-1)$, $n \geq 3$. Once again an exact formula, this time for the Kernels of the Bochner-Riesz partial summation operators, was needed in the proof.

The purpose of this paper is to show that the exact formulas can either be dispensed with (for the restriction theorem) or generalized (for the maximal theorem), thereby allowing us to obtain similar results for essentially arbitrary smooth hypersurfaces. The quality of the theorem, as measured by how large p is in the restriction theorem and how small it is in the maximal theorem, turns out to be a function of the number of principal curvatures which are nonzero at each point of the hypersurface. This is shown in Sections 2 and 3. In Section 4 we revert to the hypothesis of nonzero Gaussian curvature, but now allow the surfaces to vary smoothly from point to point and behave only asymptotically like dilations. Applications to wave equations in three space variables are given in Section 5.

Other approaches to results related to those of Section 4 have been carried out independently by Coifman, El-Kohen, Meyer, Nagel, Stein and Wainger in unpublished form.

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