

Admissibility Criteria for Hyperbolic Conservation Laws

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1. Introduction. We are concerned with a system of n hyperbolic conservation laws

$$(1.1) \quad u_t + f(u)_x = 0,$$

depending on a single space variable x , $-\infty < x < \infty$, where $u(x, t) = (u_1, u_2, \dots, u_n)$ and $f(u) = (f_1(u), \dots, f_n(u))$ are n dimensional vectors. Here $f(u)$ is a C^2 function with Jacobian matrix $f'(u)$. The matrix $f'(u)$ has real and distinct eigenvalues $\lambda_1(u), \dots, \lambda_n(u)$ with corresponding right eigenvectors $r_1(u), \dots, r_n(u)$, respectively. We let $l_1(u), \dots, l_n(u)$ be the basis of left eigenvectors of $f'(u)$ dual to $r_1(u), \dots, r_n(u)$. Thus $l_i(u) \cdot r_j(u) = \delta_{ij}$ where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

It is well known that solutions of (1.1) are not smooth in general so one looks for weak solutions in the distributional sense. These solutions contain shock waves $x = x(t)$ which travel at finite speeds $s = dx/dt$. Letting $u_{\pm} = u(x(t) \pm, t)$ be the values of $u(x, t)$ on the right and left of the shock, it follows from (1.1) that (s, u_+, u_-) satisfy the jump condition

$$(1.2) \quad s(u_+ - u_-) = f(u_+) - f(u_-).$$

Equation (1.2) is known as the Rankine-Hugoniot condition [7].

One of the main features of the theory of hyperbolic conservation laws is that weak solutions with given initial data are not necessarily unique. To guarantee uniqueness an additional criterion is required to select admissible solutions. Such a criterion is usually called an entropy condition since it is motivated by the second law of thermodynamics for gas dynamics.

We consider the set of solutions (s, u_+) of (1.2) for a fixed u_- . It is possible to describe these solutions rather precisely when u_+ is close to u_- , [1], [7]. In fact for a fixed $u_0 \in R^n$ let $S(u_0)$ be the set

$$(1.3) \quad S(u_0) = \{u : \sigma(u - u_0) = f(u) - f(u_0), \text{ scalar } \sigma = \sigma(u_0, u)\}.$$

In a small neighborhood $N(u_0)$ of u_0 , $S(u_0)$ consists of n smooth branches $S_1(u_0), \dots, S_n(u_0)$ through u_0 corresponding to the n eigenvalues $\lambda_1(u_0), \dots, \lambda_n(u_0)$. For $u \in S_i(u_0)$ the tangent vector to $S_i(u_0)$ at u is close to $r_i(u_0)$