

Local Class Field Theory via Lubin-Tate Groups

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Let k be a field complete with respect to a discrete valuation, with a finite residue class field. Let O_k be the ring of integers in k , let π be a prime element in O_k , and let $q = p^r = (O_k : \pi O_k)$, the number of elements in the residue class field. Let T be the maximal unramified extension of k .

Lubin and Tate have given a beautiful construction of a field L_π , which is abelian and totally ramified over k . Using local class field theory they showed that $T \cdot L_\pi$ is the maximal abelian extension of k . Our goal here is to give a direct proof of this fact, our Theorem 1, viewing it as a local analog of the Kronecker-Weber Theorem.

Such a proof was already given several years ago by Leopoldt, apparently along lines quite similar to ours, but has not been published. Recently, proofs have also been described by Lubin and by Rosen.

Our main result is

Theorem 1. *If F/k is abelian, then $F \subseteq T \cdot L_\pi$.*

In addition to the content and notation of [LT] and some ordinary ramification theory, we'll need a few facts about the upper numbering of ramification groups. These are the basic formulas, Herbrand's Theorem, the upper numbering of infinite Galois groups, and the theorem of Hasse-Arf: The upper jump numbers of a finite abelian extension of k are integers. This material may be found in Serre, [S].

Let U_k denote the units of O_k and $U_k^{(m)}$, the units of the form $1 + \pi^m \cdot O_k$. For any Galois group G , the upper and lower numbering of the ramification subgroups will, in the usual manner, be written G^x , G_y .

Lemma 1. *Identify $\text{Gal}(L_{\pi,m}/k)$ with $U_k/U_k^{(m)}$. Then for any n such that $0 \leq n < m$, we have $\mu \in U_k^{(n)} \setminus U_k^{(n+1)}$ iff μ is in the $(q^n - 1)$ -st ramification group (lower numbering) of $L_{\pi,m}/k$ and not in the q^n th. The $(q^n - 1)$ -st ramification group modulo the q^n th is isomorphic to $U_k^{(n)}/U_k^{(n+1)}$, which has order $q - 1$ or q as $n = 0$ or $n > 0$.*

Proof. $\mu \in O_k^*$, $\mu = 1 + a\pi^n$, $a \in O_k^*$, and let π_m denote a primitive π^m th torsion element in the Lubin-Tate module. Then