

# *Toeplitz Operators Associated with Subalgebras of the Disk Algebra*

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**I. Introduction.** For  $1 \leq p \leq \infty$ , let  $L^p$  and  $H^p$  be the usual Lebesgue and Hardy spaces of functions defined on the unit circle of the complex plane. For  $\varphi$  in  $L^\infty$ , the Toeplitz operator with symbol  $\varphi$  is the linear operator  $T_\varphi$  mapping  $H^2$  to  $H^2$  defined by  $T_\varphi(f) = P(\varphi f)$ . Here  $P$  denotes the orthogonal projection of  $L^2$  onto  $H^2$ .

The invertibility criterion for such operators was given by Devinatz and by Widom. (See [3].)

**Theorem 1.**  $T_\varphi$  is invertible if and only if both

- (a)  $\text{ess inf } |\varphi| > 0$ , and
- (b) there is an  $f$  in  $H^\infty$  with  $f^{-1}$  in  $H^\infty$  such that  $\text{Re}(\varphi f) \geq \delta$  for some  $\delta > 0$ .

The proof that these conditions are sufficient uses general considerations about Toeplitz operators including the fact that the space  $H^2$  is the closure in  $L^2$  of an algebra. The fact that these conditions are also necessary appears to be more specific and to be related to the function theoretic structure of the Hardy space. The function theoretic component in the proof of Theorem 1 is a result of Helson and Szegő which solves a problem in prediction theory. Generalizations of Theorem 1 have involved generalizations of the prediction theory result. ([1], [6], [7].) Attempts to generalize the invertibility criterion to contexts in which function theoretic considerations do not apply have produced results similar to, but less sharp than, Theorem 1. ([2].)

In this paper we give an invertibility criterion for Toeplitz operators defined on certain subspaces of  $H^2$  which are also the closure of algebras. Let  $A$  be the disk algebra; that is, the continuous functions in  $H^\infty$  and let  $A^{-1}$  denote the invertible elements of  $A$ . Let  $A_J$  be any subalgebra of  $A$  which contains the constants and which has finite codimension in  $A$ . Two examples of such subalgebras are  $\{f \text{ in } A : f'(0) = 0\}$  and  $\{f \text{ in } A : f(0) = f(1/2)\}$ . Let  $A_J^{-1}$  be the set of  $f$  in  $A_J$  with  $f^{-1}$  in  $A_J$ ; let  $H_J^2$  be the closure in  $H^2$  of  $A_J$ ; and let  $H_J^\infty = H_J^2 \cap H^\infty$ .

We wish to define and study Toeplitz operators on the spaces  $H_J^2$ . Given  $\varphi$  in  $L^\infty$  we define the operator  $R_\varphi$  mapping  $H_J^2$  to itself by  $R_\varphi(f) = P_J(\varphi f)$ . Here  $P_J$  denotes the orthogonal projection of  $L^2$  onto  $H_J^2$ . We will also wish