

A Quadratic Integral Equation, II

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Introduction. We are concerned here with the nonlinear integral equation

$$(0.1) \quad u(x) = f(x) + \lambda + \int_x^1 u(y)u(y-x)dy, \quad 0 \leq x \leq 1.$$

Here $f(x)$ is a given real-valued integrable function, λ is real, and a real-valued integrable function $u(x)$ is sought which satisfies (0.1) for almost all x in $[0,1]$. If $f(x)$ is extended to be even with support in $[-1,1]$, $u(x)$ is a solution of (0.1) which is extended to be equal zero outside $[0,1]$ and $\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x)e^{i\xi x} dx$ denotes the Fourier transform of a function g , it is not hard to prove that

$$(0.2) \quad 1 - \lambda \hat{f}(\xi) = |1 - \lambda \hat{u}(\xi)|^2, \quad \xi \text{ real.}$$

A classical result of B. Ja. Levin and M. G. Krein [5] asserts that conversely, if f is as above and $1 - \lambda \hat{f}(\xi) \geq 0$ for all real ξ , there exists a real-valued integrable function u which satisfies (0.2), has support in $[0,1]$ and consequently satisfies equation (0.1).

We are concerned in this paper with refinements of the Levin-Krein result. It is proved in [7] that if $f|_{[0,1]}$ is C^k , any solution $u|_{[0,1]}$ is C^k . Furthermore, there is an interval $[\lambda_-, \lambda_+]$ such that (0.1) has a real-valued, integrable solution if and only if $\lambda_- \leq \lambda \leq \lambda_+$. If f is not identically zero, $\lambda \in [\lambda_-, \lambda_+]$ and $\lambda \neq 0$, equation (0.1) has infinitely many solutions. However, there is a unique real-valued solution $u = u_\lambda$ such that $1 - \lambda \hat{u}(z) \neq 0$ for all complex numbers z such that $\text{Im}(z) > 0$. If f is continuous we have remarked that $u_\lambda|_{[0,1]}$ is continuous, and it is proved in [7] that the map $\lambda \rightarrow u_\lambda \in C[0,1]$ is continuous for $\lambda_- \leq \lambda \leq \lambda_+$. In fact the latter continuity property, especially continuity at $\lambda = 0$, is another way of uniquely specifying u_λ . We shall study the solutions u_λ in this paper, although other solutions of (0.1) are also of interest; see [7].

Our motivation comes from several sources. The factorization given in (0.2) has proved useful in solving some nonlinear integral equations from mathematical physics; see [3], [4], [10] and [11]. A number of problems here remain open, and it is our hope that some of the results and techniques introduced in [7] and here may shed light on the equations discussed, for example, in [4]. Independently of mathematical physics, we believe that