

Weak Compactness in $L_\infty(\mu, X)$

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Probably everything that is known about weakly compact subsets of $L_\infty(\mu)$ for a finite measure μ can be proved by an easy technique originally suggested by Figiel. The idea is to let W be a weakly compact subset of $L_\infty(\mu)$ and to use the Factorization Theorem to find a reflexive Banach space R and an operator $T: R \rightarrow L_\infty(\mu)$ such that $W \subset T(B_R)$ (here B_R is the closed unit ball of R). The next step is to consider the adjoint $T^*: L_1 \rightarrow R^*$ and realize that T^* is representable as a Bochner integral. As consequences of the (norm) measurability of the kernel of T^* , one finds that W is separable, that weakly convergent sequences in $L_\infty(\mu)$ converge almost everywhere and that, for each $\varepsilon > 0$, there is a set E whose complement has μ -measure less than ε such that $\chi_E \cdot W$ is norm compact in $L_\infty(\mu)$. Recently, Khurana [9] has shown that if X is a Banach space with the Schur property (i.e., weakly convergent sequences converge in norm), then weakly convergent sequences in $L_\infty(\mu, X)$ converge almost everywhere in the norm (of X).

Khurana's theorem immediately suggests the question of whether Figiel's argument can be adapted to study weak compactness in $L_\infty(\mu, X)$ for a Banach space X with the Schur property. The authors are happy to report that the answer is yes. Before we go on, let us fix some terminology and notation.

Throughout this paper (Ω, Σ, μ) is a finite measure space, and X and Y are Banach spaces with respective duals X^* and Y^* . The space $B(X, Y)$ is the space of all bounded linear operators from X to Y and $K(X, Y)$ is the subspace of $B(X, Y)$ consisting of the compact operators. The space $L_\infty(\mu, X)$ is the space of all essentially bounded measurable functions $f: \Omega \rightarrow X$ under the norm

$$\|f\|_\infty = \text{ess sup } \|f\|_X.$$

The first two results are the underpinning for succeeding theorems of this paper. The first is a little-known lemma which was recently isolated by Pettis [11].

Lemma 1. (Pettis) *Let Y be any Banach space with the Radon-Nikodym property and suppose X is separable. If $G: \Sigma \rightarrow B(X, Y)$ is a vector measure such that $\|G(E)\|_{B(X, Y)} \leq \mu(E)$ for all E in Σ , then there exists a bounded function $g: \Omega \rightarrow B(X, Y)$ such that $g(\cdot)(x)$ is Bochner integrable for all x in X and*