

# *Elliptic Isoperimetric Problems*

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**Introduction.** In this paper, we consider the problem of finding a surface,  $S$ , minimizing the integral of an integrand  $\Phi$  over  $S$  subject to the requirements that  $S$  have a given boundary and that the integral of another integrand  $\Psi$  over  $S$  equal a specified value,  $\psi$ . Such a problem is called an *isoperimetric problem*. We work in the context of rectifiable and integral currents and elliptic integrands.

The isoperimetric problem arises naturally as a generalization of the basic problem of the calculus of variations. The earliest such problem, from which the name is derived, is that of enclosing the largest area with a curve of given length. This problem motivates the example presented in 1.7(1). There we employ an idea due to R. Gulliver to define a positive, elliptic integrand,  $\Phi$ , that is appropriate for the problem. Another problem of isoperimetric type is that of finding the curve formed when a chain, attached at its ends to two fixed points, hangs under its own weight. Here the length is given and the gravitational potential energy is to be minimized. This problem dates back to Euler (see [WR; pages 133–138]). We consider a general problem, motivated by this example, in 1.7(2). There is an interesting difficulty here, since the gravitational potential, in a constant gravitational field, is not bounded below. We deal with this using a method suggested by E. Giusti's treatment of the pendent water drop (see [GE]).

In treating the existence question for the isoperimetric problem, we find that a compatibility condition must be imposed on  $\Phi$ ,  $\Psi$ , and  $\psi$ ; an example to show this is given in 1.2(1). It appears the condition must be global. In 1.3 we give a condition on  $\psi$  sufficient for the existence of a solution. The condition is that  $\psi$  lie between the absolute minimum for the  $\Psi$  integral subject to the boundary condition, denoted by  $\psi_0$ , and the minimum over surfaces for which the first variation of the  $\Phi$  integral vanishes, denoted by  $\psi_1$ . The condition seems natural, but it is not easy to give general properties of  $\Phi$  and  $\Psi$  which imply  $\psi_0 < \psi_1$ . Indeed, much of this paper is devoted to showing that if  $\Psi$  is the two-dimensional area integrand on  $\mathbf{R}^3$  and  $\Phi$  is a sufficiently anisotropic constant coefficient ellipsoidal integrand of degree two on  $\mathbf{R}^3$ , then for an open dense set of boundaries  $\psi_0 < \psi_1$  holds (see 2.6). Ellipsoidal integrands were introduced in another context by J. Taylor (see [TJ]). For a pair of specific integrands, it is often possible to conclude that  $\psi_0 < \psi_1$  holds by applying 1.6.