

# A Note on Commutators

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**Introduction.** In this paper we will consider commutators of singular integrals. The commutator operators we consider are of the type  $[b, L](f)(x) = b(x)Lf(x) - L(bf)(x)$ .  $L$  will be the fractional integral operator [6, page 116]. The function  $b(x)$  will be a BMO function on  $\mathbb{R}^n$ . We shall study the  $L^p$  boundedness properties of these operators and will seek to characterize BMO functions using these operators. Commutators have appeared in [1] and [3]. In [3] using the Fefferman-Stein duality of [4] a factorization of  $H^1$  functions was obtained. To do so BMO commutators were considered and in particular  $L$  was taken to be a Calderón-Zygmund operator. To obtain the necessary estimates for fractional integrals we will introduce a generalized version of a maximal operator which appeared in [5]. This maximal function enables us to prove the requisite distributional inequality. A factorization theorem for  $H^1$  involving fractional integrals is readily deducible from our theorem, using the arguments of [3].

**Definitions and statement of results.** The fractional integral operator  $I_\alpha$  is defined as

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad n \geq 1, \quad 0 < \alpha < n.$$

We denote the Hardy-Littlewood maximal function of  $f$  by  $Mf(x)$ . We also need the maximal function  $f_{\alpha,p}^*(x)$ .

$$f_{\alpha,p}^*(x) = \sup_{x \in Q} \left( \frac{1}{|Q|^{1-\alpha p/n}} \int_Q |f(y)|^p dy \right)^{1/p} \quad (\text{see [5]})$$

where  $Q$  is a cube centered at  $x$ , sides parallel to the axes. A function  $b(y) \in L_{\text{loc}}^1(\mathbb{R}^n)$  is said to be in  $\text{BMO}(\mathbb{R}^n)$  if and only if

$$\sup_Q \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy < \infty$$

where  $b_Q = (1/|Q|) \int_Q b(y) dy$ .

The BMO norm of  $b(y)$  will be denoted by  $\|b\|_*$  and is given by

$$\|b\|_* = \sup_Q \frac{1}{|Q|} \int_Q |b - b_Q| dy.$$