

# *Classification of Spaces by Fixed Point Properties*

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**1. Introduction.** A space is said to have the fixed point property (fpp) if each self-map of the space has a fixed point. In this paper we start from an idea of H. Hopf and consider how close a space comes to having the fpp, even when it does not have the fpp. We restrict ourselves to Hausdorff spaces and continuous functions.

The basic idea is the following definition, which states that under certain conditions a topological space has the fpp with respect to another topological space.

**Definition 1.1.**  $X$  has the  $Y$ -fpp (or  $X \mathcal{F} Y$ ) if for each pair of maps  $f: X \rightarrow X$ ,  $g: X \rightarrow Y$  there exists some  $x \in X$  such that  $g(x) = gf(x)$ . It will follow easily that  $X$  has the fpp if and only if it has the fpp with respect to all spaces.

Hopf [1] defined a map  $g: X \rightarrow Y$  to be *free* if there exists a self-map  $f: X \rightarrow X$  such that for each  $x \in X$ ,  $g(x) \neq gf(x)$ . The slightly different formulation given above was used by Worosz [5]. Hopf showed in [1] that if  $X$  is compact and connected then  $X \mathcal{F} R^1$ . His main result in this paper is that if  $X$  is a unicoherent, locally connected continuum and  $Y$  is one-dimensional, then  $X \mathcal{F} Y$ . On sphere maps he showed that  $S^n$  does not have the  $R^n$ -fpp ( $S^n \not\mathcal{F} R^n$ ) for  $n > 1$  and asked for the largest  $k$  such that  $S^n \mathcal{F} R^k$ . This question was also asked by Krasnoselski [3] and was answered later by Pannwitz [4] who showed by examples that  $k = 1$  for any  $n$ . This contrasts with the Borsuk-Ulam Theorem which states that if  $\alpha: S^n \rightarrow S^n$  is the antipodal map and  $g: S^n \rightarrow E^n$  is any map then there exists  $x \in S^n$  such that  $g(x) = g\alpha(x)$ . Thus if the antipodality condition on  $\alpha$  is omitted the theorem is false not only for  $E^n$  but also for any Euclidean space of dimension greater than one.

If  $X$  has the fpp we say that  $X \in \mathcal{F}$ . The connection between the  $Y$ -fpp and the fpp is given here.

**Proposition 1.2.** *The following statements are equivalent.*

- (a)  $X \in \mathcal{F}$
- (b)  $X \mathcal{F} Y$  for all non-empty spaces  $Y$
- (c)  $X \mathcal{F} X$ .