

Local Analytic Isometric Deformations

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1. Consider a Riemannian manifold U which is realized isometrically as a submanifold of a Euclidean space E , $f: U \rightarrow E$. Let p be some point of U . An isometric deformation of $f(U)$ in a neighborhood of p consists of an open set V in U which contains p , an open interval I of the real line which contains zero, and a smooth map $F: V \times I \rightarrow E$ such that for each $t \in I$ the map $V \rightarrow E$ is isometric and for $t = 0$ this map coincides with the restriction of f to V . Set $n = \dim U$ and $N = \dim E$. For N not too large with respect to n , one expects a generic submanifold to have no isometric deformations except those induced by the rigid motions of E . (These isometric deformations are said to be trivial.) For N large with respect to n , one expects a generic submanifold to have non-trivial isometric deformations. It was essentially proved in [2] that isometric deformations exist for $N > (1/2)n(n+1) + n$ and f non-degenerate.

A variant of this was considered in [3]. Let H be a codimension one submanifold of U passing through p . One seeks isometric deformations of $f(U)$ which leave $f(H)$ unchanged. That is, the map $F: V \times I \rightarrow E$ is to satisfy $F(x, t) = f(x)$ for $x \in H \cap V$ and $t \in I$. In [3] it was shown that if such a deformation exists when $N \leq (1/2)n(n+1)$ then H must be an asymptotic hypersurface (for the definition, see below). And it was also shown that if $N \geq (1/2)n(n+1) + n + 1$ then, again when f is non-degenerate, such deformations are possible for any H . In that paper it was conjectured that for f non-degenerate and N in the range $(1/2)n(n+1) < N \leq (1/2)n(n+1) + n$ such deformations should be possible whenever H is non-asymptotic. The conjecture was proved for $n = 2$ by reducing the problem to one of hyperbolic equations. (Indeed several things suggest that isometric embedding problems can be handled using the theory of hyperbolic partial differential equations.) Here we prove the conjecture in general; see Theorem 3.

Isometric deformation problems are clearly related to the isometric embedding problem. Here the standard result for analytic Riemannian manifolds is the Cartan-Janet Theorem: If U is a real analytic Riemannian manifold of dimension n then each point $p \in U$ has some neighborhood V which can be embedded by a real analytic and isometric map into the Euclidean space E of dimension $n(n+1)/2$.

The number $n(n+1)/2$ is sometimes called the Cartan-Janet dimension. The differential equations which govern isometric embeddings into E^N are determined,