

Non-uniqueness for a Semilinear Initial Value Problem

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1. Introduction. In this paper we study the semilinear partial differential equation

$$(1.1) \quad \partial_t \psi(t, x) = \Delta \psi(t, x) + |\psi(t, x)|^{\gamma-1} \psi(t, x),$$

where $(t, x) \in (0, \infty) \times \mathbf{R}^n$, $\Delta = \sum (\partial/\partial x_i)^2$ acts only on the space variables, and $\gamma > 1$ is a fixed parameter. Our basic point of view is to regard $\psi(t) = \psi(t, \cdot)$ as a curve in $L^p(\mathbf{R}^n)$ or $C_0(\mathbf{R}^n)$, the continuous functions vanishing at infinity. The initial value problem associated with (1.1) has been studied in the L^p setting in [14], [15], and [16]. The questions which have been investigated include existence and uniqueness of local solutions, existence and non-existence of global solutions, blowing-up of solutions in finite time, and non-existence of local solutions. Here, we focus on the question of uniqueness of solutions; and as a result of our techniques, we also have something to say about the existence of global solutions.

The main result of this paper is that under certain conditions involving p , n , and γ , there exists a non-trivial global solution of (1.1) in

$$C([0, \infty); L^p(\mathbf{R}^n)) \cap C^1((0, \infty); L^p(\mathbf{R}^n))$$

such that $\psi(0) \equiv 0$. Thus we get at least three different solution curves, ψ , $-\psi$, and 0, emanating from the initial data 0. Moreover, the particular ψ we construct is strictly positive for $t > 0$. Also, we prove a uniqueness theorem which shows that the non-uniqueness result is in some sense optimal. Our final result concerning (1.1) is the existence of non-trivial global solutions to (1.1) under conditions involving n and γ which prohibit the existence of global non-negative solutions other than the trivial one. (See Lemma 3.1 in [2] or Section 1 in [16].)

To place this perhaps surprising non-uniqueness result in the context of previous knowledge, it is helpful to recall some known results concerning existence and uniqueness of solutions to the initial value problem associated with (1.1). (See [14] and [15].) The most common technique in the study of such initial value problems is to consider the non-linear term as a perturbation by using the "variation of parameters" integral equation,

$$(1.2) \quad \psi(t) = e^{t\Delta} f + \int_0^t e^{(t-s)\Delta} (|\psi(s)|^{\gamma-1} \psi(s)) ds.$$