

# *The Operator $M_z$ on Hilbert Spaces of Analytic Functions*

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**I. Introduction.** Let  $\Omega$  be an open, bounded, connected subset of the complex plane whose complement consists of a finite number of simply connected sets with non-empty interiors. Let  $\mathfrak{A}^2(\Omega)$  be the Hilbert space of functions analytic and square integrable over  $\Omega$ . The operator  $M_z$ , multiplication by  $z$ , is defined and bounded on  $\mathfrak{A}^2(\Omega)$ . This paper concerns itself with showing the action of  $M_z$  on  $\mathfrak{A}^2(\Omega)$  where  $\Omega$  is other than a simple disk or annulus.

First it is shown that  $\mathfrak{A}^2(\Omega)$  can be decomposed into a finite sum of submanifolds by a set of bounded projections with each projection depending on a component in the complement of  $\Omega$ . Each such submanifold is ‘suspended’ between two weighted  $\ell_2$  spaces by bounded, one-to-one, compact, linear transformations. This is extended to suspend  $\mathfrak{A}^2(\Omega)$  between two direct sums of weighted  $\ell_2$  spaces. Finally, an operator  $\Sigma$  called the multi-fork shift is defined on both direct sums of weighted  $\ell_2$  spaces and it is shown by a commuting diagram how  $M_z$  on  $\mathfrak{A}^2(\Omega)$  is related to  $\Sigma$ .

The easiest way to show that  $\mathfrak{A}^2(\Omega)$  is a Hilbert space is to start with  $\mathcal{L}^2(\Omega)$ , the Hilbert space of functions measurable and square integrable over  $\Omega$ , whose norm is denoted by  $\| \cdot \|_{\Omega}$ . Define  $\mathfrak{A}^2(\Omega)$  as the subspace of functions analytic over  $\Omega$  and claim  $\mathfrak{A}^2(\Omega)$  is closed.

**Lemma 1.1.** *Let  $f \in \mathfrak{A}^2(\Omega)$ . If  $r > 0$  is the distance between a fixed  $z \in \Omega$  and the boundary of  $\Omega$ , then*

$$|f(z)| \leq \|f\|_{\Omega} / \sqrt{\pi} \cdot r.$$

The proof is found in Epstein [5, page 2]. It follows from this lemma that every function  $f \in \mathfrak{A}^2(\Omega)$  is bounded on closed subsets contained in  $\Omega$ . Therefore any Cauchy sequence (in the  $\mathcal{L}^2(\Omega)$  norm) of functions in  $\mathfrak{A}^2(\Omega)$  converges uniformly on all closed subsets contained in  $\Omega$ , hence to a function analytic over  $\Omega$ . This in turn shows  $\mathfrak{A}^2(\Omega)$  to be a closed manifold in  $\mathcal{L}^2(\Omega)$  and thus a Hilbert space. The complete proof is found in Epstein [5, pages 2–6].

The space  $\mathfrak{A}^2(\Omega)$  is relevant to a number of areas of research. It appears in the work done in the area of conformal mappings and the reproducing kernel functions (see Bergmann [2] and Epstein [5]) as well as in some work done in approximation