Entire Functions of Operators

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Let T be a bounded operator on a separable infinite-dimensional Hilbert space. In [2], P. R. Halmos observed that if f is an entire function and if $f(T) = 0$, then there is a polynomial p such that $p(T) = 0$. Later, P. Fillmore [1], in the course of a discussion of invariant operator ranges, showed by an ingenious argument that if there is an entire function f for which $T^* = f(T)$, then for some polynomial p, $T^* = p(T)$. A fusion of these results would say that if $aT^* + b$, for complex a and b, is an entire function of T, then it is in fact a polynomial function of T. This formulation suggests the possibility of replacing $aT^* + b$ by other polynomials in T^*.

Theorem 1. *Suppose that p is a polynomial, f is an entire function, and $p(T^*) = f(T)$. Then there is a polynomial q such that $p(T^*) = q(T)$.*

The proof will require two intermediate results: first, a general theorem concerning expansion of entire functions in terms of polynomials; second, a fact about the replacement of the operator equation $p(T^*) = f(T)$ by the complex equation $p(\bar{z}) = f(z)$ for z in the spectrum of T.

I. An expansion theorem. If f is an entire function, then a standard (and frequently useful) fact is that $f(z)$ can be written as a combination of z and functions of z^2 : $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_{2n} z^{2n} + z \sum_{n=0}^{\infty} a_{2n+1} z^{2n} = f_0(z^2) + z f_1(z^2)$, where f_0 and f_1 are entire functions. Equally easily, f can be written in terms of z, z^2, and functions of z^3 : $f(z) = g_0(z^3) + z g_1(z^3) + z^2 g_2(z^3)$. It is far from obvious that a similar expansion of f is possible in terms of functions of polynomials that are not monomials.

Theorem 2. *Let f be an entire function and let p be a polynomial of degree k. There exist entire function $g_0, g_1, \ldots, g_{k-1}$ such that*

$$f(z) = g_0(p(z)) + z g_1(p(z)) + \ldots + z^{k-1} g_{k-1}(p(z)).$$

Proof. It does no harm to assume that p is monic, say $p(z) = z^k + a_{k-1} z^{k-1} + \ldots + a_1 z + a_0$. Consider first the case $f(z) = z^n$. If $n = 0$, there is no difficulty: set $g_0^{(0)}(w) = 1$ and $g_j^{(0)}(w) = 0$ for $j \neq 0$. For each n, define functions $g_j^{(n)}$, $j = 0, 1, \ldots, k - 1$, recursively as follows:

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