

Smooth Solutions for Variational Inequalities with Obstacles

GRAHAM H. WILLIAMS

0. Introduction. Suppose $\Omega \subseteq \mathbf{R}^n$ is a bounded open set with boundary $\partial\Omega \in C^2$. Suppose $\psi_1, \psi_2 \in C^2(\bar{\Omega})$ and $\phi \in C^2(\partial\Omega)$ satisfy $\psi_1 \leq \phi \leq \psi_2$ on $\partial\Omega$ and $\psi_1 \leq \psi_2$ on Ω . Suppose $a_i(x, z, p)$, $i = 1, \dots, n$ and $a(x, z, p)$ are real valued functions on $\Omega \times \mathbf{R} \times \mathbf{R}^n$. Then we denote

$$\mathbf{K} = \{v \in C^1(\bar{\Omega}) : v = \phi \text{ on } \partial\Omega, \psi_1 \leq v \leq \psi_2 \text{ on } \Omega\}$$

and if $u, v \in C^1(\bar{\Omega})$,

$$(A(u), v) = \int_{\Omega} \left\{ \sum_{i=1}^n a_i(x, u, Du) D_i v + a(x, u, Du) v \right\} dx.$$

In this paper we study the existence of smooth solutions to the variational inequality:

Find $u \in \mathbf{K}$ such that

$$(0.1) \quad (A(u), v - u) \geq 0 \quad \text{for all } v \in \mathbf{K}.$$

We also consider the similar problem when \mathbf{K} is replaced by

$$\mathbf{K}' = \{v \in C^1(\bar{\Omega}) : v = \phi \text{ on } \partial\Omega, \psi_1 \leq v \text{ on } \Omega\}.$$

Such problems arise naturally in many physical situations especially those which involve minimizing, over \mathbf{K} or \mathbf{K}' , a functional of the form

$$\int_{\Omega} F(x, u, Du) dx.$$

In this case we are led to study (0.1) with $a_i(x, z, p) = (\partial F / \partial p_i)(x, z, p)$ and $a(x, z, p) = (\partial F / \partial z)(x, z, p)$.

For further information and results the reader is referred to the survey articles by Kinderlehrer [8], Mosco [14] and Stampacchia [18] and also the recent book by Kinderlehrer and Stampacchia [9].

Associated with (0.1) is a Dirichlet problem obtained by removing the obstacles ψ_1 and ψ_2 . That is:

Find $u \in C^1(\bar{\Omega})$ such that $u = \phi$ on $\partial\Omega$ and