

Lipschitz Conditions and Quasiconformal Mappings

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1. Introduction. A mapping f of a set A in euclidean n -space R^n into R^n is said to be Hölder continuous with exponent α , $0 < \alpha \leq 1$, at a point x in A if there is a number M such that

$$(1) \quad |f(y) - f(x)| \leq M|y - x|^\alpha$$

for all y in A . Should (1) hold, with fixed M and α , for all points x and y in A , we will say that f is uniformly Hölder continuous with exponent α in A or that f belongs to the Lipschitz class with exponent α in A . We use the notation $\text{Lip}_\alpha(A)$ to designate this class.

Let D and D' be bounded domains in R^n and let f be a quasiconformal mapping of D onto D' . It is well known that f belongs to $\text{Lip}_\alpha(A)$ for each compact subset A of D with $\alpha = K_I(f)^{1/(1-n)}$, where $K_I(f)$ denotes the inner dilatation of f . In the plane this fact was demonstrated by Ahlfors [1] and by Mori [20]; the result was later extended to higher dimensions independently by Callender [4], by Gehring [9] and by Rešetnjak [29]. In general, however, f need not belong to $\text{Lip}_\alpha(D)$ for any α , even if f is continuous up to the boundary of D . It is our purpose in this paper to establish criteria which permit one to infer that f belongs to some global Lipschitz class in D . For example, we show that if f admits an extension to a continuous mapping of \bar{D} and if the Hölder condition (1) is satisfied for all points x in ∂D and y in D , then f belongs to $\text{Lip}_\beta(D)$ for $\beta = \min \{\alpha, K_I(f)^{1/(1-n)}\}$.

Particular attention will be focused on the case where either D or D' is the unit ball B^n . In this situation it should be observed that, if some quasiconformal mapping of D onto D' is uniformly Hölder continuous in D , then each such mapping enjoys the same property. This assertion follows easily from the fact that a quasiconformal self-mapping of B^n is uniformly Hölder continuous. It is not surprising, therefore, that in this setting criteria for uniform Hölder continuity reflect the intrinsic geometric properties of D or D' .

We consider first a quasiconformal mapping f of B^n onto a bounded domain D and exhibit geometric conditions on D which are sufficient to insure that f belongs to $\text{Lip}_\alpha(B^n)$ for some α . For example, given a cross-set S of D dividing D into two subdomains, we let $D^*(S)$ denote the component of $D \setminus S$ of smaller diameter