

Necessary Conditions for Hankel Multipliers

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1. Introduction. Fix $\alpha > -1/2$ and let $L^{p,\mu} = L^{p,\mu}_\alpha(0,\infty)$, $\mu \in \mathbf{R}$, $1 \leq p < \infty$, be the set of Lebesgue measurable functions with finite norm

$$\|f\|_{p,\mu} = \left(\int_0^\infty |t^\mu f(t)|^p t^{2\alpha+1} dt \right)^{1/p}.$$

When $\mu = 0$ we write $L^{p,0} = L^p$ and $\|\cdot\|_{p,0} = \|\cdot\|_p$. For $f \in L^1$ we define the modified [8] Hankel transform of order α by

$$(1.1) \quad \mathcal{H}_\alpha(f)(\rho) \equiv f^\wedge(\rho) := \int_0^\infty f(t) \mathcal{J}_\alpha(\rho t) t^{2\alpha+1} dt,$$

where $\mathcal{J}_\alpha(t) = t^{-\alpha} J_\alpha(t)$, J_α being the Bessel function of the first kind. It is well known [11], [12], [14, page 176], [16] that the Hankel transform of an L^p -function satisfies certain differentiability and/or Lipschitz conditions. The purposes of this paper are

i) to sharpen these results by deriving an appropriate analog of the classical Hausdorff-Young inequality ($1/p + 1/p' = 1$)

$$(1.2) \quad \|\mathcal{H}_\alpha(f)\|_{p'} \leq C \|f\|_p \quad (f \in L^p, 1 \leq p \leq 2),$$

and then

ii) to deduce necessary multiplier conditions by proceeding analogously to [17].

To establish an analog of (1.2) containing smoothness properties of $\mathcal{H}_\alpha(f)$ we need suitable $(L^{1,\mu_1}, L^{\infty,\mu_2})$ and $(L^{2,\mu_3}, L^{2,\mu_4})$ estimates on certain well-behaved subsets dense in $L^{p,\mu}_\alpha$, $1 \leq p \leq 2$, for all μ such that $\mu p + 2\alpha + 2 > 0$, so that we can apply the Riesz-Thorin interpolation theorem. Let us first describe these subsets $S^\wedge_0(k)$: for $k \in \mathbf{N} = \{1, 2, \dots\}$ a continuous function on $[0, \infty)$ is said to belong to $S^\wedge_0(k)$ if it satisfies the following properties

- i) it is rapidly decreasing,
- ii) it is infinitely differentiable away from the origin and,
- iii) $\mathcal{H}_\lambda(f)$ has compact support in $[0, \infty)$ for all $\lambda \in (-1/2, k]$.

Lemma 1. *If $1 \leq p \leq 2$, $\mu p + 2\alpha + 2 > 0$ and $k \in \mathbf{N}$, then $S^\wedge_0(k)$ is dense in $L^{p,\mu}_\alpha$.*