

# Extremal Quasiconformal Mappings with Angular Complex Dilatation

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**1. Introduction.** Let  $U$  denote the upper half plane and let  $H(U)$  denote the linear space of functions,  $f$ , which are analytic on  $U$  and satisfy  $\|f\| \equiv \int_U |f(z)|d\sigma(z) < \infty$ . Here, and in what follows,  $\sigma$  denotes area measure on the complex plane; also, a bounded, complex-valued, measurable function on  $U$  is called a *dilatation* and, if  $\kappa$  is a dilatation,  $\|\kappa\|_\infty \equiv \operatorname{ess\,sup}_{z \in U} |\kappa(z)|$ . In this paper we define a class of dilatations—the *angular class*. Then, we explicitly characterize the dilatations in this class which satisfy the following extremal condition:

$$(1.1) \quad \sup_{\substack{f \in H(U) \\ \|f\|=1}} \left| \int_U f(z)\kappa(z)d\sigma(z) \right| = \|\kappa\|_\infty.$$

The importance of this characterization derives from the fundamental theorem of Richard Hamilton, Edgar Reich and Kurt Strebel stated below. A quasiconformal map is *extremal* (for its boundary values) if the maximum modulus of its complex dilatation is least among all maps with the same boundary values.

**Theorem** [1], [5]. *A dilatation,  $\kappa$ , with  $\|\kappa\|_\infty < 1$ , is the complex dilatation of an extremal quasiconformal mapping of  $U$  if and only if it satisfies condition (1.1).*

Because of this theorem, our main result (Theorem 1 of Section 2) explicitly identifies the extremal quasiconformal mappings with angular complex dilatation. In light of the Hamilton-Reich-Strebel theorem, dilatations satisfying condition (1.1) shall be called *extremal* dilatations.

The angular class contains previously studied special classes, including the “piecewise continuous” dilatations of [3]. It is more important that the angular class includes all dilatations of the form  $\kappa(z) = \bar{g}(z)/|g(z)|$  where  $g$  is meromorphic on the closure of  $U$  and at  $\infty$ . For, Theorem 1 tells when such dilatations are extremal, and thereby proves the conjecture of Sethares stated in [6].

For another application, set  $S(z) = z + k\bar{z}$ ,  $0 < k < 1$ , and regard  $S$  as a quasiconformal mapping of a plane domain  $P$ . In general, it is not known if  $S$  is extremal, but Theorem 1 yields necessary and sufficient conditions if  $P$  is simply connected and bounded by lines and circles.