

Nonlinear Hodge Theory on Punctured Riemannian Manifolds

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Introduction

This paper proves a nonlinear Hodge theorem modeled on a regular Neumann problem for the global prolongation of the gas dynamics equation (with infinite sonic speed) to a Riemannian manifold with boundary and with a finite set of exterior punctures. The metric of the manifold has poles of a certain type at the punctures with respect to local geodesic radial coordinates. The manifold is of dimension three or more. The paper also contains a DeRham theorem for a class of closed differential 1-forms, with support on these punctured manifolds and with poles at the punctures. *However, the main result* of this paper is an existence, uniqueness, and continuous data dependence theorem for a nonregular Neumann problem for the gas dynamics equation (with finite sonic speed) in exterior domains of \mathbf{R}^n , $n \geq 3$. This result includes the solution of the profile problem for a time stationary, irrotational, polytropic gas flow in \mathbf{R}^3 as a special case. It is thus of considerable interest to applied gas dynamicists.

Let M be a Riemannian manifold with boundary. Let M be of dimension three or more. Let M have a finite set of interior punctures at which the metric g_{ij} of M has poles. Let there be a ball about each puncture that is isometrically diffeomorphic to the punctured unit ball in \mathbf{R}^n . Such a manifold is called a punctured manifold. We solve the regular Neumann problem below, for the prolongation of the gas dynamics equation (with infinite sonic speed) on a punctured manifold.

The regular Neumann problem. Let M be a punctured manifold. Show that there is a unique C^1 1-form ω satisfying the elliptic system:

A) (irrotationality). $d\omega = 0$

B) (conservation of mass). $\delta\rho(x, Q(\omega))\omega = 0$, where $Q(\omega) = g^{ij}\omega_i\omega_j = \|\omega\|_x^2$ and where ρ is the regular density function defined in Section I, and also the following conditions:

1. (Neumann Condition). The normal component $N\omega$ of ω vanishes at the boundary of M . That is: $N\omega = 0$ on ∂M .
2. (Vanishing Periods at the Punctures). Let Γ be a curve contractable to a puncture in the closure of M . Then