

Quadrature, Scattering, and Representations

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§1. Introduction. Let $S^2 = \{(x,y,z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the standard 2-sphere. Let H_n be the vector space of homogeneous polynomials of degree n in the three variables x, y, z .

Definition. A quadrature formula for H_n is a set of r points p_1, \dots, p_r on S^2 and constants c_1, \dots, c_r such that for all $f \in H_n$

$$(1.1) \quad \int_{S^2} f d\Omega = \sum_{j=1}^r c_j f(p_j)$$

where $d\Omega$ is the $SO(3)$ -invariant measure on S^2 of total mass 1. The p_j 's are called the *nodes of the quadrature formula*.

Of course, given n , it is most interesting to make r as small as possible. Let $r(n)$ be the minimal value of r . The following problems suggest themselves.

- (1) Find $r(n)$.
- (2) How rigid is the configuration of the nodes in a quadrature formula of length $r(n)$?

In this paper neither of these problems is solved. Concerning (1) we give only the bounds

$$(1.2) \quad \frac{(n+1)(n+2)}{2} = \dim H_n \leq r(2n) \leq \dim H_{2n} = \frac{(2n+1)(2n+2)}{2}$$

which are easily obtainable by methods in e.g. [7].

Concerning (2) we show that in degrees 2 and 4 the above lower bounds are attained giving rise to rigid configurations of nodes which are in fact vertices of the regular octahedron (degree 2) and the regular icosahedron (degree 4). We also show that although the regular dodecahedron has vertices lying on 10 lines through the origin ($10 = \dim H_3$), there is no quadrature formula of length 10 for H_6 . Details of this work may be found in Section 5. The existence of the minimal quadrature formulas referred to in Theorems 5.1 and 5.2 was known earlier [2], but I am not aware of any published proofs of the uniqueness of these formulas