

Variability Regions for Families of Univalent Functions

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1. Introduction. Let \mathcal{A} denote the set of functions that are analytic in $\Delta = \{z: |z| < 1\}$ with the topology of convergence that is uniform on compact subsets of Δ . If $\mathcal{F} \subset \mathcal{A}$ and $|\zeta| < 1$ then the n^{th} variability region $V_n(\mathcal{F}, \zeta)$ consists of the points $(w_0, w_1, \dots, w_{n-1})$ in \mathbb{C}^n so that $w_k = f^{(k)}(\zeta)$ ($k = 0, 1, \dots, n-1$) (where $f^{(0)} = f$) for some function f in \mathcal{F} . We study $V_n(\mathcal{F}, \zeta)$ for families \mathcal{F} related to univalent functions and identify the functions which correspond to points on $\partial V_n(\mathcal{F}, \zeta)$. Information about $V_n(\mathcal{F}, \zeta)$ is useful in solving extremal problems over \mathcal{F} . Indeed, knowledge about $\partial V_n(\mathcal{F}, \zeta)$ often is sufficient to obtain such solutions. We illustrate this technique by finding the form for the solutions to various extremal problems. Several of the conclusions were derived earlier by variational methods.

Convexity ideas are used to study $V_n(\mathcal{F}, \zeta)$. If \mathcal{F} is a compact, convex subset of \mathcal{A} then by considering support planes to $V_n(\mathcal{F}, \zeta)$ we find that points on $\partial V_n(\mathcal{F}, \zeta)$ are generated by suitable extreme points of \mathcal{F} . The argument is particularly effective when \mathcal{F} has a simple set of extreme points, such as $f(z) = F(xz)$ where $|x| = 1$.

We obtain information about $V_n(\mathcal{F}, \zeta)$ for a number of non-convex families \mathcal{F} which are in one-to-one correspondence with a convex family \mathcal{G} for which the argument described above can be made. The map $\mathcal{F} \rightarrow \mathcal{G}$ establishes a homeomorphism between the variability regions of the two families. This kind of transfer of information from one region to another was used by several authors recently [1], [2], [5], [8], [10] and [11].

This paper is dedicated to my wife, Billie, who was most supportive when this research was carried out.

2. General properties of variability regions for convex families. Throughout this section we assume that \mathcal{F} is a compact, convex subset of \mathcal{A} . Then $V_n(\mathcal{F}, \zeta)$ is a compact, convex subset of \mathbb{C}^n .

If $W' = (w'_1, w'_2, \dots, w'_n) \in \partial V_n(\mathcal{F}, \zeta)$ then there is a support plane Γ to $V_n(\mathcal{F}, \zeta)$ through W' . The half-space determined by Γ which contains $V_n(\mathcal{F}, \zeta)$ is described by an inequality