

Extension of Analytic Functions and the Topology in Spaces of Analytic Functions

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1. Introduction. Let X be a compact subset of the complex plane \mathbb{C} . Given a family \mathcal{F} of functions analytic on X , we would like to know when there exists a single open neighborhood of X to which every function in \mathcal{F} can be extended. Elementary considerations provide a simple necessary condition: There exists a positive number ρ_0 such that for each point x in X and each function f in \mathcal{F} , the radius of convergence of the Taylor series of f at x is at least ρ_0 . The principal result of this paper is that this necessary condition is also sufficient, provided that X has only a finite number of connected components. Although this result may not seem surprising, it is in fact false whenever X has infinitely many components. Moreover, it is false for certain compact connected subsets of \mathbb{C}^2 . (For further comments, see the concluding paragraphs of Section 2.)

Our work is motivated in part by some functional-analytic questions. Denote by $\mathcal{O}(X)$ the algebra of germs on X of functions analytic near X . For each open set U containing X , let $\mathcal{O}(U)$ denote the Fréchet algebra of analytic functions on U , equipped with the topology of uniform convergence on compact sets. We may view $\mathcal{O}(X)$ as the inductive limit of the algebras $\mathcal{O}(U)$, and endow $\mathcal{O}(X)$ with the inductive limit topology, i.e., the finest topology which renders each of the natural maps $r_U: \mathcal{O}(U) \rightarrow \mathcal{O}(X)$ continuous. Allen, Dales and McClure [1] have shown, using a general result of Komatsu [6], that this topology is in fact locally convex and thus agrees with the locally convex inductive limit topology. This topology is not metrizable, although it is determined by its convergent sequences; i.e., a subset of $\mathcal{O}(X)$ is closed if and only if it is sequentially closed. It follows from general functional-analytic considerations (see Edwards [4], for example) that a sequence $\{f_n\}$ in $\mathcal{O}(X)$ converges to 0 if and only if there are a neighborhood U of X and a sequence $\{\tilde{f}_n\}$ which converges to 0 in $\mathcal{O}(U)$ such that $r_U(\tilde{f}_n) = f_n$ for each n (so that \tilde{f}_n is an extension of f_n to U). The difficulty with this description is precisely the one with which our extension theorem is designed to deal: how to determine, directly from the sequence $\{f_n\}$ in $\mathcal{O}(X)$, when such a neighborhood U of X exists. Thus, our extension theorem leads immediately to a purely internal description of the topology of $\mathcal{O}(X)$. We also obtain results about linear functionals on $\mathcal{O}(X)$ and integral representations of functions which are analytic on $\mathbb{C} \setminus X$ and vanish at infinity.